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CONSTRUCTION OF DOUBLE SAMPLING s-CONTROL CHARTS FOR AGILE MANUFACTURING

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SUMMARY

Double sampling (DS) $\overline{X}$-control charts are designed to allow quick detection of a small shift of process mean and provides a quick response in an agile manufacturing environment. However, the DS $\overline{X}$-control charts assume that the process standard deviation remains unchanged throughout the entire course of the statistical process control. Therefore, a complementary DS chart that can be used to monitor the process variation caused by changes in process standard deviation should be developed. In this paper, the development of the DS s-charts for quickly detecting small shift in process standard deviation for agile manufacturing is presented. The construction of the DS s-charts is based on the same concepts in constructing the DS $\overline{X}$-charts and is formulated as an optimization problem and solved with a genetic algorithm. The efficiency of the DS s-control chart is compared with that of the traditional s-control chart. The results show that the DS s-control charts can be a more economically preferable alternative in detecting small shifts than traditional s-control charts. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: statistical quality control; double sampling s chart; agile manufacturing; genetic algorithm

1. INTRODUCTION

In any production process, regardless of how well designed or carefully maintained, a certain amount of inherent or natural variability or 'background noises' is always present. This variability in key quality characteristics usually arises from three sources: improperly adjusted machines, operator errors and/or defective raw materials. Such variability usually represents an unacceptable level of process performance. We refer to these sources of variability that are not a part of the chance cause pattern as 'assignable causes'. Once a process is in a state of statistical control, it keeps producing acceptable products for a relatively long period of time. However, occasionally assignable causes will occur, seemingly at random, resulting in a 'shift' to an out-of-control state where a larger proportion of the process output does not conform to the requirements. A major objective of statistical quality control is to quickly detect the occurrence of assignable causes or process shifts so that investigation of the process and corrective action may be undertaken before a large number of non-conforming units are manufactured. The control chart is an on-line process control technique widely used for this purpose.

Knowledge of the behavior of chance variations is the foundation on which control chart analysis rests. If a group of data is studied and it is found that their variation conforms to a statistical pattern that might reasonably be produced by chance causes, then it is assumed that no special assignable causes are present. The conditions which produced this variation are, accordingly, said to be under control. They are under control in the sense that, if chance causes are alone at work, then the amount and character of the variation may be predicted for large numbers and it is not possible to trace the variation of a specific instance to a particular cause. On the other hand, if the variations in the data do not conform to a pattern that might reasonably be produced by chance causes, then it is concluded that one or more assignable causes are at a work. In this case the conditions producing the variation are said to be out of control. In the case of statistical process control using control charts, any group of data outside the control limits of the control charts is taken as an out of control signal and actions need to be taken to discover and eliminate the assignable causes of the variation.

As today's manufacturing firms are moving towards agile manufacturing, quick and economic on-line

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statistical process control solutions are in high demand. Double sampling (DS) $\overline{X}$-control charts present a good alternative for statistical process control in an agile manufacturing environment. The efficiency of the DS $\overline{X}$-control chart for detecting changes in process mean has been compared with other types of control charts such as Shewhart, variable sampling intervals (VSIs), cumulative-sum (CUSUM) and exponentially weighted moving averages (EWMAs) by Daudin [1]. When being compared to a fixed sample size procedure, the DS procedure was chosen such that its average sample size when the process is in control is equal to the fixed sample size. The results of the comparison reported in [1] are summarized in the following.

In comparing the DS control chart with the VSI, the average time to signal (ATS) was used. The ATS is defined as the expected value of the time between the start of the process and the time when the chart signals. The results of the comparison showed that when the time required in collecting and measuring the sample is negligible, the DS scheme is more efficient than the VSI one.

When being compared to a classic Shewhart chart, a DS chart showed better detection of smaller to moderate shifts. Detection of large shifts is better accomplished with the Shewhart chart. Another advantage of using a DS chart over the Shewhart chart is the dramatic decrease in average sample size. When the process is in control, this decrease is nearly 50%. Based on the results of the comparison, Daudin [1] concluded that the DS $\overline{X}$-chart might be substituted for the Shewhart $\overline{X}$-chart whenever the improvement in efficiency outweighs the administration/trouble costs. In this paper, Daudin [1], defined the efficiency as the highest magnitude of the shift in process that the control chart scheme is capable of detecting while having a certain sample size.

The comparison with the combined Shewhart CUSUM chart showed that the two-sided combined Shewhart CUSUM schemes are better for detecting small shifts and the DS chart is better for detecting large shifts. The efficiency of the DS chart is close to that of the two-sided combined Shewhart CUSUM schemes with $k = 1$, but very different from the two-sided combined Shewhart CUSUM schemes with $k = 0.25$. Here, k is the reference value (or the allowance or the slack value) and is often chosen as about halfway between the target value for the mean and the out-of-control value of the mean that we are interested in quickly detecting [2]. Therefore, the DS chart may be appropriate when greater efficiency is required for small shifts than with the Shewhart chart and protection against large shifts is also important. However, when large shifts are very unusual, the CUSUM or combined Shewhart CUSUM schemes are best.

For comparison with the EWMA scheme, DS charts were designed to match average run lengths (ARL) with corresponding EWMA charts when the process is in control. When $\lambda = 0.75$, the DS chart has a greater efficiency than the EWMA chart. However, when $\lambda = 0.5$ or 0.25, the EWMA chart is better for detecting small shifts and the DS chart is better for detecting large shifts. Here $0 \leqslant \lambda \leqslant 1$ is a weighting factor. As λ decreases, the weight on previous history $(1 - \lambda)$ increases [3].

The DS $\overline{X}$-control charts assume that the process standard deviation remains unchanged throughout the whole course of the process control. Therefore, the DS $\overline{X}$-charts alone are not sufficient to monitor the process, as the process standard deviation could be shifted while the mean of the process stays the same. A change in process standard deviation with a fixed process mean could lead to a higher proportion of non-conforming parts from the process, although the process is under the control of the $\overline{X}$-control chart. For quick detection of smaller shifts in the process standard deviation, the double sampling s-control chart, which was developed in this paper, could be a good complement to the DS $\overline{X}$-control chart.

Nowadays there are a wide variety of charts for monitoring process variability. Eventually, the choice will be in favor of more economical and statistically sound at the same time. The method proposed in this paper leads to a smaller average sample size compared to traditional s-control charts, hence reducing the cost of the process monitoring. There are a few interesting papers published recently proposing new methods in economic design of control charts for variables. The work by Costa and Rahim [4] of the economic design of $\overline{X}$- and R-charts for simultaneously monitoring both the mean and variance of a continuous production process is one such paper. The product variable quality characteristic is assumed to be normally distributed and the process is subject to two independent assignable causes. One cause changes the process mean and the other changes the process variance. It is also assumed that the occurrence times of the assignable causes are described by a Weibull distribution with increasing failure rate. They developed a cost model and adopted a non-uniform sampling interval scheme. Then, by adding statistical constraints they extended the model to an economic–statistical design model for achieving desired levels of statistical performance while minimizing the expected cost. As expected, economic–statistical design is more costly than economic design because of the added constraints. The study shows that the statistical performance of the control charts with respect to the average time to signal due to false alarm can be significantly improved by an added cost.

The research work by Tolley and English [5] is a significant work in terms of studying the economic–statistical design (according to their definition, economic design of a constrained control chart). They present a comparison between the cost performances of the exponentially weighted moving average (EWMA) and the combined EWMA and $\overline{X}$-control chart schemes. In particular, they explored the impact of constraining the in-control average run length on the optimal cost performances of both schemes. This research suggests that the region of greatest chart discrimination occurs when the shift is small, and the EWMA weighting parameter, λ, is large. Examination of cost surfaces under in-control average run length constraints revealed that for the combined EWMA and $\overline{X}$-charting schemes the cost model is not a well-behaved function and presents optimization challenges. No significant differences in estimated total cost for the two charting schemes is revealed though.

Because of the efficiency of DS $\overline{X}$-charts in detecting a small shift in process mean, one could naturally extend the same principle of DS $\overline{X}$-control to develop a DS s-chart control. Up to now, no research work on the construction of DS s-chart has been reported in the literature.

The remainder of the paper is organized as follows. In Section 2, brief background information on traditional s-control charts is provided. Section 3 is devoted to the method of development for constructing the DS s-control chart. In Section 4, the use of a genetic algorithm for solving the DS s-chart construction problem is explained. In Section 5, the efficiency of the DS s-charts is compared with that of the traditional s-charts and the results of the comparison are presented. Finally, Section 6 concludes the paper.

2. BACKGROUND OF TRADITIONAL s-CONTROL CHARTS

Let s be a sample standard deviation and σ be a process standard deviation. If samples are drawn from a population with a normal probability distribution, then the mean of the sample s is given as $c_4\sigma$. It is also known that, as n increases, the distribution of s becomes closer and closer to a symmetrical distribution [3].

Theoretical knowledge of the distribution of s in samples from a normal universe is the basis for 3σ limits on the control charts for s. The sample standard deviation s is not an unbiased estimator of σ. If the underlying distribution is normal, then s actually estimates $c_4\sigma$, where c_4 is a constant that depends on the sample size n [2]. Furthermore, the standard deviation of s is $\sigma\sqrt{1-c_4^2}$. This information was used to establish a traditional control chart on s. For the case with known process standard deviation σ, since $E(S) = c_4\sigma$, the centerline for the chart is $c_4\sigma$. The control limits for an s-chart can be computed as follows:

$$UCL_S = c_4\sigma + 3\sigma\sqrt{1-c_4^2}$$
$$LCL_S = c_4\sigma - 3\sigma\sqrt{1-c_4^2}$$

where:

$$c_4 = \left(\sqrt{\frac{2}{n-1}}\right)\frac{\Gamma(n/2)}{\Gamma(n-1/2)}$$

and $\Gamma(\cdot)$ is a Gamma function.

3. DEVELOPMENT OF DS s-CONTROL CHARTS

Up to now, no effort on developing methods for constructing DS s-charts has been reported in the literature. In this section, the development of the DS s-chart is presented. Specifically, the following procedure is proposed for the chart construction. Two successive samples are taken without any intervening time, which means that the samples are coming from the same probability distribution. This could be achieved by collecting a master sample of $n_1 + n_2$ units all at the same time. First, analyze the first sample n_1 and then decide whether to analyze the remaining units. Figure 1 provides the graphical representation of the DS s-control chart. The control limits of the chart are shown in the number of the standard deviations of the estimated sample standard deviation. All the steps involved in the DS s-control chart could be summarized into a simple procedure.

Before the DS s-control chart procedure is presented, let us define the coefficient for sample size of n_1:

$$c_{41} = \left(\sqrt{\frac{2}{n_1-1}}\right)\frac{\Gamma(n_1/2)}{\Gamma[(n_1-1)/2]}$$

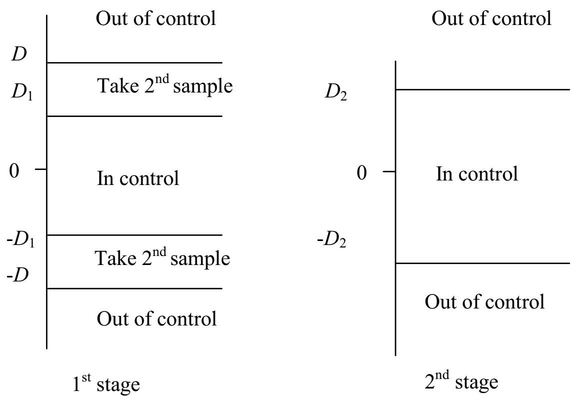

Figure 1. Graphic view of the DS s-control chart

and the coefficient for sample size of $n_1 + n_2$ (see the derivation of c_4 in Appendix A):

$$c_4 = \left(\sqrt{\frac{2}{n_1 + n_2 - 2}}\right) \frac{\Gamma[(n_1 + n_2 - 1)/2]}{\Gamma[(n_1 + n_2 - 2)/2]}$$

Then the mean of the random variable s_1 at the first stage is $c_{41}\sigma$, the standard deviation of s_1 is $\sigma\sqrt{1 - c_{41}^2}$, the mean of the random variable s_{12} at the second stage is $c_4\sigma$, the standard deviation of s_{12} is $\sigma\sqrt{1 - c_4^2}$. It is also assumed that s_1 and s_{12} are normally distributed.

DS s-control chart procedure

(1) Take an initial sample of size n_1. Calculate the standard deviation of the sample, s_1.

(2) If $(s_1 - c_{41}\sigma)\Big/\left(\sigma\sqrt{1 - c_{41}^2}\right)$ lies in the range $[-D_1, D_1]$, the process is considered to be under control.

(3) If $(s_1 - c_{41}\sigma)\Big/\left(\sigma\sqrt{1 - c_{41}^2}\right)$ lies in the range $(-\infty, -D]$ and $[D, +\infty)$ the process is considered to be out of control.

(4) If $(s_1 - c_{41}\sigma)\Big/\left(\sigma\sqrt{1 - c_{41}^2}\right)$ lies in the intervals $[-D, -D_1]$ and $[D_1, D]$, take a second sample of size n_2 and calculate the total sample standard deviation

$$s_{12} = \sqrt{\frac{\sum_{i=1}^{2}(n_i - 1)s_i^2}{\sum_{i=1}^{2} n_i - 2}} = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

(5) If $(s_{12} - c_4\sigma)\Big/\left(\sigma\sqrt{1 - c_4^2}\right)$ lies in the interval $[-D_2, D_2]$, then the process is in control, otherwise the process is out of control.

The design of a DS s-chart involves determining the values of the following five parameters:

- n_1, sample size of the first sample;
- n_2, sample size of the second sample;
- D_1 and $-D_1$, the limits on the first sample within which the process is said to be in control;
- D and $-D$, the limits on the first sample beyond which the process is said to be out of control;
- D_2 and $-D_2$, the limits on the second sample within which the process is said to be in control.

In this paper, the optimal design of a DS s-control chart is formulated as an optimization problem. The objective function of the optimization problem is to minimize the average number of samples during the normal operation of the process. The optimization problem has two constraints. The first constraint is that the probability of concluding that a normal process is out of control is less than a constant α, which is normally specified as the manufacturer's risk. The second constraint is that the probability of concluding that an abnormal process is in control is less than β, which is normally specified as the consumer's risk.

Let us define the intervals

$$I_1 = \left[c_{41}\sigma - D_1\sigma\sqrt{1 - c_{41}^2},\, c_{41}\sigma + D_1\sigma\sqrt{1 - c_{41}^2}\right]$$

$$I_2 = \left[c_{41}\sigma - D_\sigma\sqrt{1 - c_{41}^2},\, c_{41}\sigma - D_1\sigma\sqrt{1 - c_{41}^2}\right] \cup \left[c_{41}\sigma + D_1\sigma\sqrt{1 - c_{41}^2},\, c_{41}\sigma + D_\sigma\sqrt{1 - c_{41}^2}\right]$$

$$I_3 = \left(-\infty,\, c_{41}\sigma - D_\sigma\sqrt{1 - c_{41}^2}\right] \cup \left[c_{41}\sigma + D_\sigma\sqrt{1 - c_{41}^2},\, \infty\right)$$

$$I_4 = \left[c_4\sigma - D_2\sigma\sqrt{1 - c_4^2},\, c_4\sigma + D_2\sigma\sqrt{1 - c_4^2}\right]$$

$$I_5 = \left(-\infty,\, c_4\sigma - D_2\sigma\sqrt{1 - c_4^2}\right] \cup \left[c_4\sigma + D_2\sigma\sqrt{1 - c_4^2},\, +\infty\right)$$

Mathematically the optimization problem can be written as follows:

$$\min_{n_1, n_2, D, D_1, D_2} \quad n_1 + n_2 \Pr[s_1 \in I_2 \mid \sigma = \sigma_0] \qquad (1)$$

Subject to:

$$\Pr[\text{Out of Control} \mid \sigma = \sigma_0] \leqslant \alpha \qquad (2)$$

$$\Pr[\text{In Control} \mid \sigma = \sigma_1] \leqslant \beta \qquad (3)$$

The objective function (1) in the optimization model is to minimize the average sample size subject to two

Formula 1

$$P_{a1} = \Pr[s_1 \in I_1] = \int_{s_1 \in I_1} \frac{\exp\left(-\left((s_1 - c_{41}\sigma_0)\Big/\left(\sigma_0\sqrt{1-c_{41}^2}\right)\right)^2 \Big/ 2\right)}{\sqrt{2\pi}\,\sigma_0\sqrt{1-c_{41}^2}}\,\mathrm{d}s_1$$

Formula 2

$$\begin{aligned}
P_{a2} &= \Pr[s_{12} \in I_4] \\
&= \Pr\left[c_4\sigma - D_2\sigma\sqrt{1-c_4^2} \leqslant s_{12} \leqslant c_4\sigma + D_2\sigma\sqrt{1-c_4^2}\right] \\
&= \Pr\left[c_4\sigma - D_2\sigma\sqrt{1-c_4^2} \leqslant \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}} \leqslant c_4\sigma + D_2\sigma\sqrt{1-c_4^2}\right] \\
&= \Pr\left[\left(c_4\sigma - D_2\sigma\sqrt{1-c_4^2}\right)^2 \leqslant \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2} \leqslant \left(c_4\sigma + D_2\sigma\sqrt{1-c_4^2}\right)^2\right] \\
&= \Pr\left[\sigma^2\left(c_4 - D_2\sqrt{1-c_4^2}\right)^2 \leqslant \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2} \leqslant \sigma^2\left(c_4 + D_2\sqrt{1-c_4^2}\right)^2\right] \\
&= \Pr\left[\left(c_4 - D_2\sqrt{1-c_4^2}\right)^2 \leqslant \frac{(n_1-1)s_1^2/\sigma^2 + (n_2-1)s_2^2/\sigma^2}{n_1+n_2-2} \leqslant \left(c_4 + D_2\sqrt{1-c_4^2}\right)^2\right]
\end{aligned}$$

constraints. Constraint (2) ensures that the probability of making a false alarm is not greater than α (Type I error) and constraint (3) ensures that the probability of not detecting a shift in process standard deviation is not greater than β (Type II error). In addition to constraints (2) and (3), lower and upper bounds are imposed on D, D_1 and D_2. The values of the lower and upper bounds are set up as suggested by Daudin [1] for practical implementation of DS charts. Daudin [1] recommended that D must be higher than the classical values 3 or 3.09. A good choice is $D = 4$ or 5. D_1 must be lower than the classical value for the warning limit, which often is taken to be equal to 2. A good choice for D_1 is between 1.3 and 1.8. Also, integer constraints are imposed on n_1 and n_2. In order to solve the optimization problem (1)–(3), the probabilities in constraints (2) and (3) should be defined in terms of the decision variables n_1, n_2, D, D_1 and D_2. Therefore, the derivation of probabilities Pr[Out of Control $\mid \sigma = \sigma_0$] and Pr[In Control $\mid \sigma = \sigma_1$] is provided next.

Because the control chart involves two stages, the decision whether the process is in control or not should be based on analyzing both stages. Let P_{a1} and P_{a2} be the probabilities that the process is in control at the first and second stage, respectively. Then, the probability that the process is in control can be computed as $P_a = P_{a1} + P_{a2}$. Consequently, the probability that the process is out of control is equal to $1 - P_a$.

Let us first expand constraint (2). It could be represented as

$$1 - (P_{a1} + P_{a2}) \leqslant \alpha$$

When there is no shift in process standard deviation, the probability that the process is in control at the first stage is computed as shown in Formula 1.

The probability that the process is in control at the second stage when there is no shift in the process standard deviation can be derived as shown in Formula 2.

Let

$$r = n_1 + n_2 - 2$$

$$x = \frac{(n_1-1)s_1^2}{\sigma^2}, \quad y = \frac{(n_2-1)s_2^2}{\sigma^2}$$

Random variables x and y follow chi-square distributions with $n_1 - 1$ and $n_2 - 1$ degrees of freedom

Formula 3

$$1-\left\{\int_{s_1\in I_1}\frac{\exp\left(-\left((s_1-c_{41}\sigma_0)\Big/\left(\sigma_0\sqrt{1-c_{41}^2}\right)\right)^2\Big/2\right)}{\sqrt{2\pi}\,\sigma_0\sqrt{1-c_{41}^2}}\,\mathrm{d}s_1\right\}-\int_{x\in I_2^*}\left[\int_{y\in[A,B]}\frac{y^{((n_2-1)/2)-1}\,\mathrm{e}^{-y/2}}{2^{(n_2-1)/2}\Gamma((n_2-1)/2)}\,\mathrm{d}y\right]\frac{x^{((n_1-1)/2)-1}\,\mathrm{e}^{-x/2}}{2^{(n_1-1)/2}\Gamma((n_1-1)/2)}\,\mathrm{d}x\leqslant\alpha$$

respectively, i.e.

$$x\sim\chi^2_{n_1-1},\quad y\sim\chi^2_{n_2-1}$$

Therefore, the conditional probability is computed as follows:

$$\Pr[s_{12}\in I_4\mid s_1=x]=\Pr\left[r\left(c_4-D_2\sqrt{1-c_4^2}\right)^2-x\leqslant y\leqslant r\left(c_4+D_2\sqrt{1-c_4^2}\right)^2-x\right]$$

Let

$$A=r\left(c_4+D_2\sqrt{1-c_4^2}\right)^2-x$$
$$B=r\left(c_4-D_2\sqrt{1-c_4^2}\right)^2-x$$

The chi-square probability density function is

$$f(x)=\frac{1}{2^{(n_1-1)/2}\Gamma((n_1-1)/2)}x^{((n_1-1)/2)-1}\,\mathrm{e}^{-x/2},\quad x>0$$

$$f(y)=\frac{1}{2^{(n_2-1)/2}\Gamma((n_2-1)/2)}y^{((n_2-1)/2)-1}\,\mathrm{e}^{-y/2},\quad y>0$$

$$P_{a2}=\int_{x\in I_2^*}\{\Pr[s_{12}\in I_4\mid s_1=x]\}f(x)\,\mathrm{d}x=\int_{x\in I_2^*}\left[\int_{y\in[A,B]}\frac{y^{((n_2-1)/2)-1}\,\mathrm{e}^{-y/2}}{2^{(n_2-1)/2}\Gamma((n_2-1)/2)}\,\mathrm{d}y\right]\times\frac{x^{((n_1-1)/2)-1}\,\mathrm{e}^{-x/2}}{2^{(n_1-1)/2}\Gamma((n_1-1)/2)}\,\mathrm{d}x$$

where I_2^* can be written as follows (see the derivation of I_2^* in Appendix B):

$$I_2^*=\left[(n_1-1)\left(c_{41}-D\sqrt{1-c_{41}^2}\right)^2,\ (n_1-1)\left(c_{41}-D_1\sqrt{1-c_{41}^2}\right)^2\right]\cup\left[(n_1-1)\left(c_{41}+D_1\sqrt{1-c_{41}^2}\right)^2,\ (n_1-1)\left(c_{41}+D\sqrt{1-c_{41}^2}\right)^2\right]$$

Thus constraint (2) can be written as: Pr[Out of Control $\mid \sigma=\sigma_0]\leqslant\alpha$, i.e. as shown in Formula 3.

Constraint (3) ensures that the probability of deciding that there is no shift in process, when there is actual shift in process standard deviation less than or equal to a predefined acceptable probability:

$$P_{a1}+P_{a2}\leqslant\beta$$

When there is a shift in the process standard deviation from σ to σ_1, then the probability that the process is in control at the first stage is equal to

$$P_{a1}=\Pr[s_1\in I_1]=\int_{s_1\in I_1}\frac{\exp\left(-\left((s_1-c_{41}\sigma_1)\Big/\left(\sigma_1\sqrt{1-c_{41}^2}\right)\right)^2\Big/2\right)}{\sqrt{2\pi}\sigma_1\sqrt{1-c_{41}^2}}ds_1$$

When there is a shift in the process standard deviation from σ to σ_1, then the probability that the process is in control at the second stage is computed as shown in Formula 4.

Define

$$A_1=r\frac{\sigma^2}{\sigma_1^2}\left(c_4+D_2\sqrt{1-c_4^2}\right)^2-x$$
$$B_1=r\frac{\sigma^2}{\sigma_1^2}\left(c_4-D_2\sqrt{1-c_4^2}\right)^2-x$$

Thus, the conditional probability that random variable s_{12} will fall into interval I_4 given that random variable s_1 belongs to interval I_2 is computed as

$$P_{a2}=\Pr[s_{12}\in I_4\mid s_1\in I_2]=\int_{x\in I_2^{**}}\{\Pr[s_{12}\in I_4\mid s_1=x]\}f(x)\,\mathrm{d}x$$

Formula 4

$$\Pr\left[\frac{\sigma^2}{\sigma_1^2}\left(c_4 - D_2\sqrt{1-c_4^2}\right)^2 \leqslant \frac{(n_1-1)s_1^2/\sigma_1^2 + (n_2-1)s_2^2/\sigma_1^2}{n_1+n_2-2} \leqslant \frac{\sigma^2}{\sigma_1^2}\left(c_4 + D_2\sqrt{1-c_4^2}\right)^2\right]$$
$$= \Pr\left[r\frac{\sigma^2}{\sigma_1^2}\left(c_4 - D_2\sqrt{1-c_4^2}\right)^2 - x \leqslant y \leqslant r\frac{\sigma^2}{\sigma_1^2}\left(c_4 + D_2\sqrt{1-c_4^2}\right)^2 - x\right]$$

Formula 5

$$\int_{s_1\in I_1} \frac{\exp\left(-\left((s_1 - c_{41}\sigma_1)\Big/\left(\sigma_1\sqrt{1-c_{41}^2}\right)\right)^2 \Big/ 2\right)}{\sqrt{2\pi}\,\sigma_1\sqrt{1-c_{41}^2}}\,\mathrm{d}s_1 + \int_{x\in I_2^{**}}\left[\int_{y\in[A_1,B_1]} \frac{y^{((n_2-1)/2)-1}\,\mathrm{e}^{-y/2}}{2^{(n_2-1)/2}\Gamma((n_2-1)/2)}\,\mathrm{d}y\right]\frac{x^{((n_1-1)/2)-1}\,\mathrm{e}^{-x/2}}{2^{(n_1-1)/2}\Gamma((n_1-1)/2)}\,\mathrm{d}x \leqslant \beta$$

where I_2^{**} can be written as follows (see the derivation of I_2^{**} in Appendix B):

$$I_2^{**} = \left[(n_1-1)\frac{\sigma^2}{\sigma_1^2}\left(c_{41} - D\sqrt{1-c_{41}^2}\right)^2, (n_1-1)\frac{\sigma^2}{\sigma_1^2}\left(c_{41} - D_1\sqrt{1-c_{41}^2}\right)^2\right]$$
$$\cup \left[(n_1-1)\frac{\sigma^2}{\sigma_1^2}\left(c_{41} + D_1\sqrt{1-c_{41}^2}\right)^2, (n_1-1)\frac{\sigma^2}{\sigma_1^2}\left(c_{41} + D\sqrt{1-c_{41}^2}\right)^2\right]$$

Formula 6

$$\underset{n_1,n_2,D,D_1,D_2}{\mathrm{Min}}\; n_1 + n_2 \int_{s_1\in I_2} \frac{\exp\left(-\left((s_1 - c_{41}\sigma_0)\Big/\left(\sigma_0\sqrt{1-c_{41}^2}\right)\right)^2 \Big/ 2\right)}{\sqrt{2\pi}\,\sigma_0\sqrt{1-c_{41}^2}}\,\mathrm{d}s_1 \tag{4}$$

Subject to

$$1 - \left\{\int_{s_1\in I_1} \frac{\exp\left(-\left((s_1 - c_{41}\sigma_0)\Big/\left(\sigma_0\sqrt{1-c_{41}^2}\right)\right)^2 \Big/ 2\right)}{\sqrt{2\pi}\,\sigma_0\sqrt{1-c_{41}^2}}\,\mathrm{d}s_1\right\} - \int_{x\in I_2^{*}}\left[\int_{y\in[A,B]} \frac{y^{((n_2-1)/2)-1}\,\mathrm{e}^{-y/2}}{2^{(n_2-1)/2}\Gamma((n_2-1)/2)}\,\mathrm{d}y\right]\frac{x^{((n_1-1)/2)-1}\,\mathrm{e}^{-x/2}}{2^{(n_1-1)/2}\Gamma((n_1-1)/2)}\,\mathrm{d}x \leqslant \alpha \tag{5}$$

$$\int_{s_1\in I_1} \frac{\exp\left(-\left((s_1 - c_{41}\sigma_1)\Big/\left(\sigma_1\sqrt{1-c_{41}^2}\right)\right)^2 \Big/ 2\right)}{\sqrt{2\pi}\,\sigma_1\sqrt{1-c_{41}^2}}\,\mathrm{d}s_1 + \int_{x\in I_2^{**}}\left[\int_{y\in[A_1,B_1]} \frac{y^{((n_2-1)/2)-1}\,\mathrm{e}^{-y/2}}{2^{(n_2-1)/2}\Gamma((n_2-1)/2)}\,\mathrm{d}y\right]\frac{x^{((n_1-1)/2)-1}\,\mathrm{e}^{-x/2}}{2^{(n_1-1)/2}\Gamma((n_1-1)/2)}\,\mathrm{d}x \leqslant \beta \tag{6}$$

$$= \int_{x \in I_2^{**}} \left[\int_{y \in [A_1, B_1]} \frac{y^{((n_2-1)/2)-1}\, e^{-y/2}}{2^{(n_2-1)/2}\Gamma((n_2-1)/2)}\, dy \right] \times \frac{x^{((n_1-1)/2)-1}\, e^{-x/2}}{2^{(n_1-1)/2}\Gamma((n_1-1)/2)}\, dx$$

Constraint (3) can be written as (for a given intended change in process standard deviation)

$$\Pr[\text{In Control} \mid \sigma = \sigma_1] \leqslant \beta$$

i.e. as shown in Formula 5.

Then, the optimization problem (1)–(3) can be written as shown in Formula 6.

4. SOLVING THE OPTIMIZATION PROBLEM USING A GENETIC ALGORITHM (GA)

As one can see that the minimization problem formulated by model (4)–(6)) is characterized by mixed continuous-discrete variables and discontinuous and non-convex solution space. Therefore, if standard nonlinear programming techniques are used for solving this type of optimization problem, they will be inefficient and computationally expensive. Genetic algorithms (GAs) are well suited for solving such problems and, in most cases, find a global optimum solution with a high probability [6]. Another motivation in using GAs to solve model (4)–(6) is that since the pioneering work by Holland [7], GAs have been developed into a general and robust method for solving all kinds of optimization problems (e.g. Goldberg [8], Portieter and Stander [9], Pham and Pham [10], Vinterbo and Ohno-Machado [11]) and computing software for applications of GAs have been made commercially available in the market. Thus solving model (4)–(6) using GAs provides a practical way for real-time statistical process control implementation. The GAs have been used for the statistical design of DS and triple sampling $\overline{X}$-control charts [12].

The tool used for implementing the GAs in solving model (4)–(6) is a commercially available GA software called Evolver [13]. Evolver is used as an add-in program to the Microsoft Excel spreadsheet application. The optimization model (4)–(6) is set up in an Excel spreadsheet and solved by the GA in Evolver. The operation of the genetic algorithm involves the following steps:

(a) create a random initial solution;
(b) evaluate fitness, i.e. the objective function that minimizes the average sample size when the process is in control;
(c) reproduction and mutation;
(d) generate new solutions.

The quality of the solutions generated by the GAs depends on the setup of its parameters such as population size, crossover and mutation probability. During the implementation, the values of these parameters are setup to obtain the best results.

Crossover probability determines how often crossover will be performed. If there is no crossover, an offspring (new solution) will be an exact copy of the parents (old solutions). If there is a crossover, an offspring (new solution) is made from parts of parents' chromosome. If crossover probability is 100%, then offspring are made by crossover. Crossover is made in the hope that new chromosomes will have the good parts of the old chromosomes and maybe will be better. However, it is good to leave some part of the population to survive to the next generation.

Mutation probability determines how often parts of the chromosomes will be mutated. If there is no mutation, offspring will be taken after crossover (or copy) without any change. If mutation probability is 100%, the whole chromosome is changed. Mutation is made to prevent the search by the GA falling into local extremes, but it should not occur very often, because then the GA will in fact change to random search.

In our computational experiment, the population size is set to 1000. The crossover probability is setup to 0.5 and the mutation probability is 0.06.

The integrations in (5) and (6) are computed with numerical integration using Simpson's rule [14].

5. PERFORMANCE EVALUATION OF THE DS s-CONTROL CHART

In order to evaluate the performance of the developed DS s-chart, its efficiency (measured by the average number of samples inspected to detect a shift of a certain magnitude) was compared with that of the traditional one. For any control chart for variables, the average number of samples before a shift of a certain magnitude is detected should be as small as possible. The average number of samples before the false alarm should be as large as possible. The ability of the s-charts to detect a shift in process quality is described by their operating-characteristic (OC) curves.

Another way to evaluate the decisions regarding sample size and sampling frequency is through the average run length (*ARL*) of the control chart. Essentially, *ARL* is the average number of points that must be plotted before a point indicates an out-of-control condition [2]. For any Shewhart control chart, *ARL* can be expressed as $ARL = 1/P$ (one point plots out of control) or $ARL_0 = 1/\alpha$ for the in-control *ARL* and $ARL_1 = 1/(1-\beta)$ for the out-of-control *ARL*.

In order to compare the DS s-control chart with the traditional one a procedure similar to that used by Daudin [1] for comparing DS $\overline{X}$-control carts with Shewhart charts is used. In Daudin's procedure, the efficiency of a DS $\overline{X}$-chart is compared with that of a corresponding Shewhart $\overline{X}$-chart that is matched with the same ARL_0 and ARL_1 as the DS $\overline{X}$-chart. The efficiency is measured by the number of samples collected on average to detect the shift of a given magnitude.

In our procedure, for a given pair of ARL_1 points and a given shift in process standard deviation, a DS s-chart and traditional s-chart were constructed. Then, the efficiency of the two charts was compared. Here a shift in process standard deviation is expressed as a ratio of the shifted process standard deviation σ_1 to the normal process standard deviation σ_0:

$$\lambda = \frac{\sigma_1}{\sigma_0}$$

In our computational experiment, the DS s-charts and traditional s-charts were constructed over a range of λ values from 1.2 to 6.0. The two ARL points used in the comparison were: $ARL_0 = 370.4$ ($\alpha = 0.0027$) and $ARL_1 = 1.222$ ($\beta = 0.1817$).

Since for a traditional s-chart with 3σ control limits, $ARL_0 = 370.4$, hence we try to determine its sample size n such that $ARL_1 \leqslant 1.222$, i.e.

$$\beta = \Pr\{LCL \leqslant s \leqslant UCL \mid \sigma = \sigma_1\} \leqslant 0.1817$$

For constructing a corresponding DS s-control chart, the two ARL points (α and β values) and the given σ_1 value (obtained from given λ and known σ_0) were used to solve the optimization model (4)–(6) with a GA.

It should be noticed that after the determination of two matched charts, for shifts less than λ the DS chart has a lower ARL and for shifts larger than λ the DS has a higher ARL. For example, a plot of the ARL of a traditional s-chart with a sample size of 18 and the corresponding ARL of a DS s-chart with an average sample size $E(N) = 13.68$ is shown in Figure 2.

From Figure 2 we see that when $\lambda \leqslant 1.8$ the DS is more efficient and when $\lambda \geqslant 1.8$ the traditional chart is more efficient.

Having constructed the charts, the efficiency of the charts is compared in terms of the number of samples required to detect a shift of a certain magnitude. The results are provided in Table 1.

From Table 1, one can observe that the results could be grouped into two major patterns. In comparison with the traditional s-control chart, the computational results show that for a relatively small shift in process standard deviation ($1.2 \leqslant \lambda \leqslant 2.1$) there is a significant reduction in the average sample size when the DS s-control charts are used. For those who are interested in the detection of shifts less than the corresponding λ, when λ changes from 1.2 to 2.0, the DS s-control chart could be used to significantly reduce the cost associated with collecting and inspecting samples.

When the shift ratio λ is greater than 2.1, the computational results show that the DS s-control chart is required to have the same sample size of $n_1 = 6$ and $n_2 = 16$ to detect those shifts, while for traditional s-control charts the sample size required to detect certain shift is decreasing as shift magnitude is increasing. The reason for such a behavior is that when the sample size is relatively small ($n_1 \leqslant 5$), the s-control chart becomes one-sided and the normal approximation is not valid anymore. Indeed, when the chart is one-sided the area under the normal distribution curve becomes less than 1. While sample size decreases the area under the normal distribution curve decreases accordingly. With probability less than unity, the normal approximation cannot provide satisfaction of the constraints. In particular, for constraint (2), there is no such a combination of design variables which could ensure the probability of a false signal less than 0.0027, because the difference between unity and the combined probability of mistakenly deciding that process is out of control for both stages (which is less than 1 by more than 0.0027) is always going to be more than 0.0027.

For the case where $\lambda < 1$ the model behaves as expected, i.e. a larger average sample size is needed to detect a smaller shifts and smaller average sample size is needed to detect larger shifts. The results also show that for the same shift, the DS s-charts require a smaller average sample size.

From the statistical theory, when the samples come from a normal population we know the expected mean and standard deviation of s in samples, but not the type of distribution. What kind of distribution does s have for small sample sizes? For example, for samples of size five or less while setting up the limits for traditional s-chart, the lower control limit is set up to zero, because $c_4\sigma - 3\sigma\sqrt{1 - c_4^2}$ is becoming negative. The reason is that the real distribution differs from the normal distribution significantly, that is why the LCL implied to be 3σ away from the mean becomes a negative number (the s distribution does not extend for 3σ from the left side of the mean, it starts less

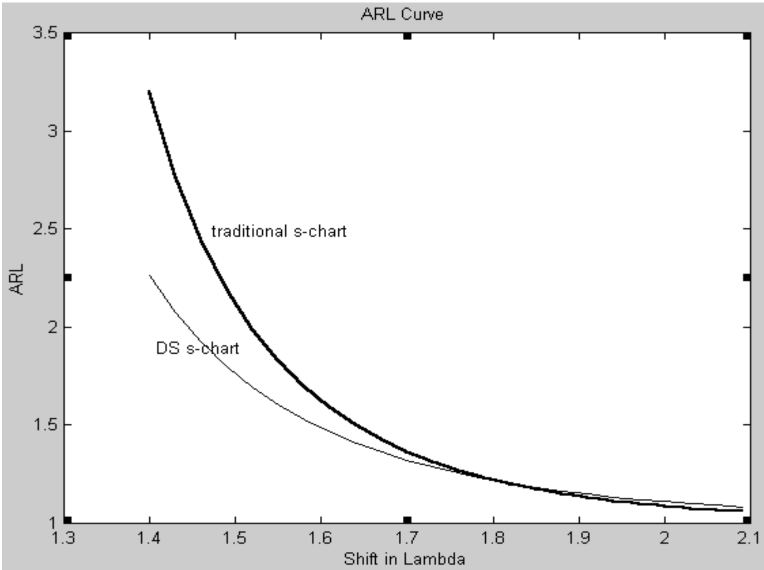

Figure 2. The *ARL* of a traditional s-control chart with a sample size of 18 and the corresponding *ARL* of a DS s-control chart with an average sample size $E(N) = 13.68$

Table 1. The results of comparison between the DS s-control charts and the Shewhart type s-control charts

	DS chart								
n	n_1	n_2	$\lambda = \sigma_1/\sigma_0$	ARL_0	ARL_1	D_1	D	D_2	$E(N)$
41	20	30	0.6	370.4	1.222	1.69	4.10	2.59	22.71
175	82	174	0.8	370.4	1.222	1.58	4.15	2.62	101.73
211	133	200	1.2	370.4	1.222	1.79	3.60	2.95	146.92
99	65	80	1.3	370.4	1.222	1.79	3.30	2.99	70.55
59	37	71	1.4	370.4	1.222	1.71	3.04	2.53	42.25
40	25	54	1.5	370.4	1.222	1.75	3.03	2.64	29.17
30	17	50	1.6	370.4	1.222	1.66	3.01	2.73	21.76
22	15	26	1.7	370.4	1.222	1.80	3.01	2.65	16.81
18	11	29	1.8	370.4	1.222	1.67	3.00	2.63	13.68
15	10	20	1.9	370.4	1.222	1.79	3.03	2.61	11.40
13	8	23	2.0	370.4	1.222	1.76	3.02	2.58	9.75
12	8	12	2.1	370.4	1.222	1.79	3.02	2.66	8.84
10	6	16	2.2	370.4	1.222	1.49	3.04	2.68	8.11
9	6	16	2.3	370.4	1.222	1.77	3.02	2.52	6.89
9	6	16	2.4	370.4	1.222	1.77	3.02	2.52	6.89
8	6	16	2.5	370.4	1.222	1.77	3.02	2.52	6.89
5	6	16	3.0	370.4	1.222	1.77	3.02	2.52	6.89
5	6	16	4.0	370.4	1.222	1.77	3.02	2.52	6.89
4	6	16	4.5	370.4	1.222	1.77	3.02	2.52	6.89
4	6	16	5.0	370.4	1.222	1.77	3.02	2.52	6.89

than 3σ away from the mean). It means that the real distribution is asymmetric, with a shorter tail from the left of the mean and a longer tail from the right.

The assumption that s has a normal distribution is becoming unconvincing as the sample size decreases, because the real distribution becomes more asymmetrical. On the other hand, we can assume that s has a normal distribution for moderate or large sample sizes because it is known that the real distribution becomes symmetrical. A good indicator of whether we can assume the normality of the distribution of s could be the LCL of the s-control chart. If it is positive, it means that the left tail extends for at least 3σ from the mean. If LCL is negative, it means that the s distribution is highly asymmetric and the smaller the sample size the more asymmetric the distribution. For samples of size five or less, it is not reasonable to assume that s has a normal distribution, because the mean is more shifted towards the left side. It means that the computational data for the shifts of more than 2.1 are not valid and cannot be used for a comparison of the two charts.

We can conclude that the assumption of the normality of s is valid only for samples of size six and more and we consider the samples of size five and less as having another distribution (non-symmetrical); we investigate this distribution more closely in other research. Also, it could be a good topic of investigation to find out whether the loosing constraint (3) could give an economical advantage due to inspecting smaller sample sizes or not.

6. CONCLUSIONS

DS $\overline{X}$-control charts are designed to allow the quick detection of a small shift of process mean and provides a quick response in an agile manufacturing environment. However, DS $\overline{X}$-control charts assume that the process standard deviation remains unchanged throughout the entire course of the statistical process control. Therefore, a complementary DS chart that can be used to monitor the process variation caused by changes in process standard deviation should be developed. In this paper, the development DS s-chart for quickly detecting small shifts in process standard deviation for agile manufacturing is presented. The construction of the DS s-charts is based on the same concepts in constructing DS $\overline{X}$-charts and is formulated as an optimization problem and solved with a GA. The efficiency of DS s-control charts is compared with that of the traditional s-control charts and the results obtained are presented. The results of the comparison show that the DS s-control charts can be a more economically preferable alternative in detecting small shifts than traditional s-control charts.

The work presented in this paper was an attempt to develop a quick and efficient control chart for the detection of a small shift in the process variance. The results of the comparison show that the average sample size required for detection of a small shift is much less than for DS s-control charts. However, the assumption that s has normal distribution is becoming invalid when sample size is less than five. That is why in this paper we did not consider the comparison of the two charts for smaller sample sizes. A future research work could include developing DS s-charts without the assumption of a normal distribution.

It is known that many practitioners still prefer R-control charts to s or to s^2, but for moderate values of sample size n, say $n \geqslant 10$, a R-chart rapidly loses its efficiency, as it ignores all the information in the sample between maximum and minimum values [2]. The results of the comparison between the DS s-control charts and the traditional s-control charts have shown that the DS s-charts can be very efficient for a sample size greater than ten. So our conviction is that the DS s-control charts could be used when the sample size is moderate or large.

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APPENDIX A. DERIVATION OF c_4

Implicitly it is being assumed that the quality measurement follows a normal distribution with a mean μ and a standard deviation σ. As random variables

$$x = \frac{(n_1 - 1)s_1^2}{\sigma^2} \quad \text{and} \quad y = \frac{(n_2 - 1)s_2^2}{\sigma^2}$$

follow chi-square distributions with degrees of freedom $n_1 - 1$ and $n_2 - 1$, respectively, the sum of two independent chi-squares is a chi-square distribution with degrees of freedom

$$(n_1 - 1) + (n_2 - 1) = n_1 + n_2 - 2$$

It now follows that

$$\begin{aligned} s_{12} &= \sqrt{\frac{(n_1-1)s_1^2+(n_2-1)s_2^2}{n_1+n_2-2}} \\ &= \sqrt{\frac{\sigma^2}{n_1+n_2-2}\left[\frac{(n_1-1)s_1^2}{\sigma^2}+\frac{(n_2-1)s_2^2}{\sigma^2}\right]} \\ &= \sqrt{\frac{\sigma^2}{n_1+n_2-2}Y} \\ &= \frac{\sigma}{\sqrt{n_1+n_2-2}}Y^{1/2} \end{aligned}$$

where Y has a chi-square distribution with n_1+n_2-2 degrees of freedom. Thus,

$$\begin{aligned} E(s) &= \mu_s \\ &= \frac{\sigma}{\sqrt{n_1+n_2-2}}E[Y^{1/2}] \\ &= \frac{\sigma}{\sqrt{n_1+n_2-2}} \\ &\quad\times\int_0^\infty y^{1/2}\frac{y^{((n_1+n_2-1)/2)-1}\,\mathrm{e}^{-y/2}}{\Gamma((n_1+n_2-1)/2)2^{(n_1+n_2-1)/2}}\,\mathrm{d}y \\ &= \sigma\frac{\Gamma((n_1+n_2-1)/2)2^{(n_1+n_2-1)/2}}{\sqrt{n_1+n_2-2}\Gamma((n_1+n_2-2)/2)2^{(n_1+n_2-1)/2}} \\ &\quad\times\int_0^\infty y^{1/2}\frac{y^{((n_1+n_2-1)/2)-1}\,\mathrm{e}^{-y/2}}{\Gamma((n_1+n_2-1)/2)2^{(n_1+n_2-1)/2}}\,\mathrm{d}y \\ &= \left(\sqrt{\frac{2}{n_1+n_2-2}}\right)\frac{\Gamma[(n_1+n_2-1)/2]}{\Gamma[(n_1+n_2-2)/2]}\sigma \\ &= c_4\sigma \end{aligned}$$

APPENDIX B. DERIVATION OF INTERVALS OF I_2^* AND I_2^{**}

Since s_1 is in interval I_2 and

$$\begin{aligned} I_2 = &\left[c_{41}\sigma - D_\sigma\sqrt{1-c_{41}^2},\, c_{41}\sigma - D_1\sigma\sqrt{1-c_{41}^2}\right] \\ &\cup\left[c_{41}\sigma + D_1\sigma\sqrt{1-c_{41}^2},\, c_{41}\sigma + D_\sigma\sqrt{1-c_{41}^2}\right] \end{aligned}$$

then

$$c_{41}\sigma - D_\sigma\sqrt{1-c_{41}^2} \leqslant s_1 \leqslant c_{41}\sigma - D_1\sigma\sqrt{1-c_{41}^2}$$

and

$$c_{41}\sigma + D_1\sigma\sqrt{1-c_{41}^2} \leqslant s_1 \leqslant c_{41}\sigma + D\sigma\sqrt{1-c_{41}^2}$$

For the case when there is no shift in the process variability, i.e. $\lambda = 1$, by multiplying both sides of the inequality signs by $(n_1-1)/\sigma^2$, we obtain

$$\begin{aligned} &\frac{(n_1-1)\left(c_{41}\sigma - D\sigma\sqrt{1-c_{41}^2}\right)^2}{\sigma^2} \\ &\quad\leqslant \frac{s_1^2(n_1-1)}{\sigma^2} \\ &\quad\leqslant \frac{(n_1-1)\left(c_{41}\sigma - D_1\sigma\sqrt{1-c_{41}^2}\right)^2}{\sigma^2} \end{aligned}$$

and

$$\begin{aligned} &\frac{(n_1-1)\left(c_{41}\sigma + D_1\sigma\sqrt{1-c_{41}^2}\right)^2}{\sigma^2} \\ &\quad\leqslant \frac{s_1^2(n_1-1)}{\sigma^2} \\ &\quad\leqslant \frac{(n_1-1)\left(c_{41}\sigma + D\sigma\sqrt{1-c_{41}^2}\right)^2}{\sigma^2} \end{aligned}$$

Since x is defined as $x = (n_1-1)s_1^2/\sigma^2$ then

$$\begin{aligned} &(n_1-1)\left(c_{41} - D\sqrt{1-c_{41}^2}\right)^2 \\ &\quad\leqslant x \leqslant (n_1-1)\left(c_{41} - D_1\sqrt{1-c_{41}^2}\right)^2 \end{aligned}$$

and

$$\begin{aligned} &(n_1-1)\left(c_{41} + D_1\sqrt{1-c_{41}^2}\right)^2 \\ &\quad\leqslant x \leqslant (n_1-1)\left(c_{41} + D\sqrt{1-c_{41}^2}\right)^2 \end{aligned}$$

Thus, interval I_2^* can be written as

$$\begin{aligned} I_2^* = &\left[(n_1-1)\left(c_{41} - D\sqrt{1-c_{41}^2}\right)^2,\right. \\ &\quad\left.(n_1-1)\left(c_{41} - D_1\sqrt{1-c_{41}^2}\right)^2\right] \\ &\cup\left[(n_1-1)\left(c_{41} + D_1\sqrt{1-c_{41}^2}\right)^2,\right. \\ &\quad\left.(n_1-1)\left(c_{41} + D\sqrt{1-c_{41}^2}\right)^2\right] \end{aligned}$$

When there is a shift in process standard deviation from σ to σ_1, then the interval I_2^{**} within which the random variable x should vary, could be obtained as follows. We have

$$c_{41}\sigma - D_\sigma\sqrt{1-c_{41}^2} \leqslant s_1 \leqslant c_{41}\sigma - D_1\sigma\sqrt{1-c_{41}^2}$$

and

$$c_{41}\sigma + D_1\sigma\sqrt{1-c_{41}^2} \leqslant s_1 \leqslant c_{41}\sigma + D\sigma\sqrt{1-c_{41}^2}$$

By multiplying by both sides of the inequality signs by $(n_1 - 1)/\sigma_1^2$, we obtain

$$\frac{(n_1-1)\left(c_{41}\sigma - D\sigma\sqrt{1-c_{41}^2}\right)^2}{\sigma_1^2} \leqslant \frac{s_1^2(n_1-1)}{\sigma_1^2} \leqslant \frac{(n_1-1)\left(c_{41}\sigma - D_1\sigma\sqrt{1-c_{41}^2}\right)^2}{\sigma_1^2}$$

and

$$\frac{(n_1-1)\left(c_{41}\sigma + D_1\sigma\sqrt{1-c_{41}^2}\right)^2}{\sigma_1^2} \leqslant \frac{s_1^2(n_1-1)}{\sigma_1^2} \leqslant \frac{(n_1-1)\left(c_{41}\sigma + D\sigma\sqrt{1-c_{41}^2}\right)^2}{\sigma_1^2}$$

Since x is defined as $x = (n_1 - 1)s_1^2/\sigma_1^2$ then

$$(n_1-1)\frac{\sigma^2}{\sigma_1^2}\left(c_{41} - D\sqrt{1-c_{41}^2}\right)^2 \leqslant x \leqslant (n_1-1)\frac{\sigma^2}{\sigma_1^2}\left(c_{41} - D_1\sqrt{1-c_{41}^2}\right)^2$$

and

$$(n_1-1)\frac{\sigma^2}{\sigma_1^2}\left(c_{41} + D_1\sqrt{1-c_{41}^2}\right)^2 \leqslant x \leqslant (n_1-1)\frac{\sigma^2}{\sigma_1^2}\left(c_{41} + D\sqrt{1-c_{41}^2}\right)^2$$

Therefore, I_2^{**} can be written as

$$I_2^{**} = \left[(n_1-1)\frac{\sigma^2}{\sigma_1^2}\left(c_{41} - D\sqrt{1-c_{41}^2}\right)^2, (n_1-1)\frac{\sigma^2}{\sigma_1^2}\left(c_{41} - D_1\sqrt{1-c_{41}^2}\right)^2\right] \cup \left[(n_1-1)\frac{\sigma^2}{\sigma_1^2}\left(c_{41} + D_1\sqrt{1-c_{41}^2}\right)^2, (n_1-1)\frac{\sigma^2}{\sigma_1^2}\left(c_{41} + D\sqrt{1-c_{41}^2}\right)^2\right]$$

REFERENCES

1. Daudin JJ. Double sampling X-bar charts. *Journal of Quality Technology* 1992; **24**(2):78–87.
2. Montgomery DC. *Introduction to Statistical Quality Control*. John Wiley and Sons: New York, NY, 2001.
3. Grant EL, Leavenworth RS. *Statistical Quality Control*. McGraw-Hill: New York, NY, 1996.
4. Costa AFB, Rahim MA. Economic design of X-bar and R charts under Weibull shock models. *Quality and Reliability Engineering International* 2000; **16**(2):143–156.
5. Tolley GO, English JR. Economic design of constrained EWMA and combined EWMA-X bar control schemes. *IIE Transactions* 2001; **33**(6):429–436.
6. Rao SS. *Engineering Optimization: Theory and Practice*. John Wiley and Sons: New York, NY, 1996.
7. Holland JH. *Adaptation in Natural and Artificial Systems*. University of Michigan Press: Ann Arbor, MI, 1975.
8. Goldberg DE. *Genetic Algorithms in Search, Optimization, and Machine Learning*. Addison-Wesley: Reading, MA, 1989.
9. Potgieter E, Stander N. Genetic algorithm applied to stiffness maximization of laminated plates: Review and comparison. *Structural Optimization* 1998; **15**(3–4):221–229.
10. Pham DT, Pham PTN. Artificial intelligence in engineering. *International Journal of Machine Tools & Manufacture* 1999; **39**(6):937–949.
11. Vinterbo S, Ohno-Machado L. Genetic algorithm approach to multidisorder diagnosis. *Artificial Intelligence in Medicine* 2000; **18**(2):117–132.
12. He D, Grigoryan A, Sigh M. Design of double and triple-sampling $\overline{X}$ control charts using genetic algorithms. *International Journal of Production Research* (accepted for publication in October 2001).
13. Palisade Corporation. *Evolver: The Genetic Algorithm Super Solver*. Palisade Corporation: Newfield, NY.
14. Gerald CF, Wheatley PO. *Applied Numerical Analysis*. Addison-Wesley: Reading, MA, 1999.

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