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Design of double- and triple-sampling X-bar control charts using genetic algorithms

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As today's manufacturing firms are moving towards agile manufacturing, quick and economic on-line statistical process control solutions are in high demand. Multiple sampling X-bar control charts are such an alternative. They can be designed to allow quick detection of a small shift in process mean and provide a quick response in an agile manufacturing environment. In this paper, the designs of double-sampling (DS) X-bar control charts are formulated and solved with a genetic algorithm. Based on the results in solving the DS chart design problems, triple sampling (TS) X-bar control charts are developed. The efficiency of the TS charts is compared with that of the DS charts. The results of the comparison show that TS charts are more efficient in terms of minimizing the average sample size.

1. Introduction

In any manufacturing processes, variability may occasionally be present in the output of the process. This variability in key quality characteristics usually arises from three sources: improperly adjusted machines, operator errors and/or defective raw materials. Such variability usually represents an unacceptable level of process performance. These sources of variability which are not part of the chance cause pattern are referred to as 'assignable causes'. Once a process is in a state of statistical control, it keeps producing acceptable products for relatively long periods. However, occasionally assignable causes will occur, seemingly at random, resulting in a 'shift' to an out-of-control state where a larger proportion of the process output does not conform to the requirements. A major objective of statistical quality control is to detect quickly the occurrence of assignable causes or process shifts so that investigation of the process and corrective action may be undertaken before a large number of non-conforming units are manufactured. The control chart is an on-line process control technique widely used for this purpose.

As today's manufacturing firms are moving towards agile manufacturing, quick and economic on-line statistical process control solutions are in high demand. Multiple sampling X-bar control charts are such an alternative. One successful application of the multiple-sampling X-bar control charts is the use of a double-sampling (DS) control chart in replacement of traditional control charts for on-line statistical process control in industries. A DS X-bar chart is quicker and more sensitive than the traditional Shewhart X-bar chart in detecting the mean shift in

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a process. In its simplest form, a DS control chart involves only two sampling stages. Although the DS X-bar control charts have been well studied for statistical process control, the power of the sampling X-bar control charts with more than two stages in statistical process control has not yet been fully exploited. The major reason is that the current methods used in constructing the DS X-bar control chart are cumbersome. If the number of sampling stages increases to more than two, it becomes more difficulty to construct the sampling X-bar control charts and investigate their properties.

In this paper, new methods for constructing DS and triple-sampling (TS) X-bar control charts based on a genetic algorithm (GA) are developed. The method provides an efficient way to construct the DS and TS X-bar control charts for agile manufacturing and provides a basis for developing a method for constructing multiple-sampling X-bar control charts.

The remainder of the paper is organized as follows. In Section 2, brief background information on traditional Shewhart X-bar control chart and DS X-bar control charts is provided. Section 3 is devoted to the methods for constructing the DS and TS X-bar control charts. In Section 4, computational results in comparing the efficiency of TS with DS charts are provided. Section 5 concludes the paper.

2. Backgrounds on the Shewhart X-bar and DS X-bar control charts

This section gives a brief review on the traditional Shewhart X-bar control chart and DS X-bar control chart.

2.1. *Shewhart X-bar control chart*

The general model of the X-bar control chart was first proposed by Walter A. Shewhart (Shewhart 1931, Montgomery 1997), and it now bears his name. Let x be a sample statistic that measures a process characteristic used to control a manufacturing process. Suppose that the mean of x is μ and the standard deviation (SD) is σ. The following three parameters that characterize the Shewhart X-bar control chart are: $\mathrm{UCL} = \mu + k\sigma$; $\mathrm{CL} = \mu - k\sigma$, where UCL is the upper control limit, CL is the centre line or the process average, LCL is the lower control limit and k is the distance of the control limits from the centreline and is expressed as a multiple of σ. The most common multiple SD used is 3. Figure 1 shows an example of what a Shewhart X-bar control chart model looks like.

The standard Shewhart X-bar chart is widely used in the industry for its simplicity. The chart is however slow in detecting small shifts in a process. In general, increasing the sample size of the Shewhart X-bar control charts improves their ability to detect shifts and drifts without increasing the occurrence of false alarms. However, increasing the sample size not only results in increased inspection costs, but also increases reaction time because a large amount of time has to be spent on the inspection of these large sample sizes. A large amount of effort has therefore been made to improve the detection capability of the Shewhart X-bar chart without increasing the sample size. Exponential Weighted Moving Average (EWMA) charts (Crowder 1987), Combined Shewhart CUSUM charts (Lucas 1982), Variable Sampling Intervals (VSI) charts (Reynolds *et al.* 1988) and Variable Sample Size (VSS) charts (Costa 1994) are examples of such modified charts.

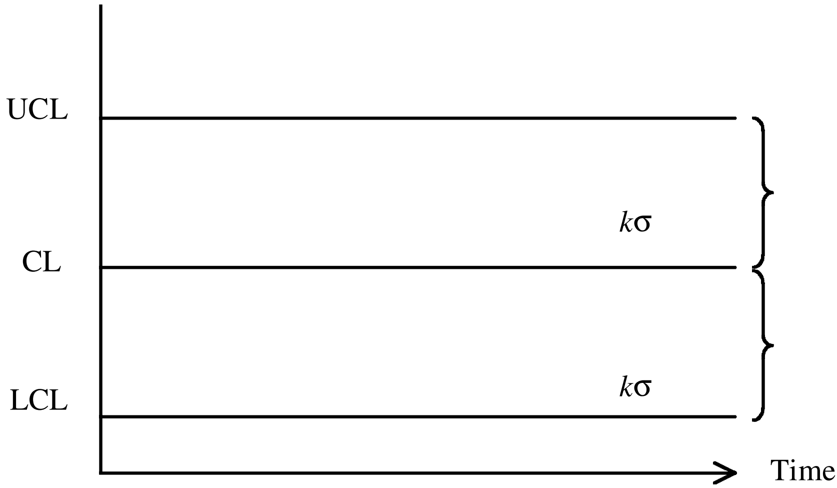

Figure 1. Shewhart X-bar control chart.

2.2. *DS X-bar control chart*

First studied by Croasdale (1974) and Daudin (1992), the DS X-bar chart offers better statistical efficiency (in terms of the average run length) than the Shewhart X-bar chart without increased sampling. Alternatively, it can reduce the sampling without reducing the statistical efficiency. In addition, DS X-bar chart offers certain advantages over other modified Shewhart X-bar charts. For example, in contrast to the VSI charts, a DS chart assumes that two successive samples can be taken without any intervening time. Thus, it is possible to use a DS chart in a practical situation where a varying time interval might be required to analyse or measure the sample. The overall advantage of the DS charts compared with other control charts is that a DS chart is determined by five parameters and, therefore, there is a great deal of flexibility in designing the chart (Daudin 1992). The efficiency of the DS X-bar control chart for detecting changes in the mean has been compared with other types of control charts such as Shewhart, VSI, CUSUM and EWMA by Daudin (1992). When being compared with a fixed sample size procedure, the DS procedure was chosen such that its average sample size when the process is in control (i.e. when $\mu = \mu_0$) is equal to the fixed sample size. The results of the comparison reported in Daudin (1992) are summarized below.

In comparing DS control charts with VSI, the average time to signal (ATS) was used. ATS is defined as the expected time between the start of the process and the time when the chart signals. The assumptions of the VSI and DS charts are different. The former assumes that the minimum time between samples is positive and can be different, while the latter assumes the minimum time between samples is negligible. The results of the comparison showed that when the time required to collect and measure the samples is negligible, the DS scheme is more efficient than the VSI one.

When being compared with a classic Shewhart chart, a DS chart was constructed with an average run length (ARL) curve matching two points on the ARL curve of the classic Shewhart chart and the minimum average sample size when the process is in control. In statistical process control, ARL is normally defined as the expected number of samples taken before the shift is detected. It was concluded that the DS chart was better at detecting moderate shifts. Detection of large shift was better accomplished with the Shewhart chart. Another advantage of using a DS chart over the Shewhart chart is the dramatic decrease in sample size. When the process is in control, this decrease is nearly 50%. Based on the results of the comparison, Daudin (1992) concluded that the DS X-bar chart might be substituted for the Shewhart X-bar chart whenever the improvement in efficiency out-weights the administration trouble/costs.

A DS X-bar chart is a procedure in which, under certain circumstances, a second sample is required before a decision can be made about the process. Daudin proposed the following procedure for constructing a DS X-bar chart (Irianto and Shinozaki 1998).

The design of a DS X-bar chart involves determining five parameters.

- n_1 = Sample size of the first sample.
- n_2 = Sample size of the second sample.
- L_1 and $-L1$ = limits on the first sample within which the process is said to be in control.
- L_2 and $-L_2$ = limits on the second sample within which the process is said to be in control.
- L and $-L$ = limits on the first sample beyond which the process is said to be out of control.

2.2.1. *Daudin's DS X-bar control procedure*

(1) Take an initial sample of size n_1. Calculate the mean of the sample $\overline{X_1}$.

(2) If $\frac{\overline{X_1} - \mu_0}{\sigma/\sqrt{n_1}}$ lies in the range $[-L_1, L_1]$, the process is in control.

(3) If $\frac{\overline{X_1} - \mu_0}{\sigma/\sqrt{n_1}}$ lies in the range $(-\infty, -L]$ or $[L, +\infty)$, the process is out of control.

(4) If $\frac{\overline{X_1} - \mu_0}{\sigma/\sqrt{n_1}}$ lies in the range $(-L, -L_1)$ or (L_1, L), take a second sample of size n_2 and calculate the sample mean $\overline{X_2}$.

(5) Calculate the total sample mean $\overline{Y} = \frac{n_1\overline{X_1} + n_2\overline{X_2}}{n_1 + n_2}$ at the second stage.

(6) If $-L < \frac{\overline{X_1} - \mu_0}{\sigma/\sqrt{n_1}} < -L_1$ or $L_1 < \frac{\overline{X_1} - \mu_0}{\sigma/\sqrt{n_1}} < L$ and if $\frac{\overline{Y} - \mu_0}{\sigma/\sqrt{n_1 + n_2}} < -L_2$ or $\frac{\overline{Y} - \mu_0}{\sigma/\sqrt{n_1 + n_2}} > L_2$, then the process is out of control, else the process is in control.

A view of the DS X-bar chart is given by figure 2.

The optimal design of a DS X-bar control chart is to determine the parameters n_1, n_2, L, L_1 and L_2, so that the average sample size is minimized. Daudin first formulated the DS X-bar control chart design problem as an optimization problem with an objective function of minimizing the average sample size (Irianto and

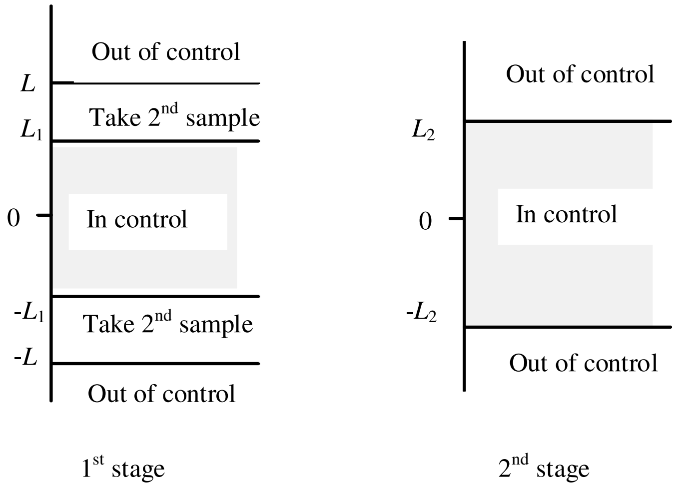

Figure 2. Graphic view of the DS X-bar chart.

Shinozaki 1998). The formulation has two constraints. The first is to ensure that the probability of concluding that the process is out of control when the process is in control is equal to α, i.e. the manufacturer's risk. The second constraint is to ensure that the probability of concluding that the process is in control when the process is out of control is equal to β, i.e. the consumer's risk. To solve the optimization problem, Daudin (1992) proposed the following approximation procedure.

2.2.2. *Daudin's approximation procedure* (DAP)

(1) Fix n_1 and n_2.
(2) For a given L (suggested to be 4–5 times SD), both constraints are used to determine L_1 (suggested to be 1.3–1.8 times SD) and L_2.
(3) Find the optimal composition of L_1, L_2 and L that minimizes the objective function for all possible pairs of (n_1, n_2).

Although it provides a convenient way to construct a DS X-bar control chart, there are limitations associated with the application of DAP in using the DS X-bar chart for real-time control. First, DAP is an exhaustive search-based heuristic procedure. Therefore, its computational time is expensive. The computational complexity problem can be further aggravated if the sampling is to be extended into more than two stages. Second, the parameters L, L_1, and L_2 are predetermined ad hoc during the search process. Hence, this may reduce the chance of obtaining good or near optimal solutions. To overcome these limitations, a more flexible and systematic algorithm should be used. Next, a GA is applied to solve the DS X-bar control chart design problems.

3. Design of DS X-bar control chart using genetic algorithm

3.1. *DS X-bar chart design formulation*

To solve the problem of an optimal design of a DS X-bar control chart using a systematic and robust method, the optimization problem is reformulated as follows:

$$\min_{n_1,n_2,L,L_1,L_2} \quad n_1 + n_2 \,\mathrm{P_r}[\overline{X_1} \in \mathrm{I}_2 | \mu = \mu_0], \tag{1}$$

subject to

$\mathrm{P_r}$[out of control$|\mu = \mu_0] \leq \alpha$, i.e.

$$1 - \{\Phi[L_1] - \Phi[-L_1]\} - \int_{z\in I_2^*} \left\{\Phi\left[c\,L_2 - z\sqrt{\frac{n_1}{n_2}}\right] - \Phi\left[-c\,L_2 - z\sqrt{\frac{n}{n_2}}\right]\right\}\varphi(z)\,\mathrm{d}z \leq \alpha. \tag{2}$$

$\mathrm{P_r}$[in control$|\mu = \mu_1] \leq \beta$ (for a given intended shift δ), i.e.

$$\{\Phi[L_1 + \delta\sqrt{n_1}] - \Phi[-L_1 + \delta\sqrt{n_1}]\} + \int_{z\in I_2^*} \left\{\Phi\left[cL_2 + rc\delta - z\sqrt{\frac{n_1}{n_2}}\right] - \Phi\left[-cL_2 + rc\delta - z\sqrt{\frac{n_1}{n_2}}\right]\right\}\varphi(z)\,\mathrm{d}z \leq \beta \tag{3}$$

$$n_i \geq 1, \text{integer, for } i = 1, 2 \tag{4}$$

$$L \geq L_l \tag{5}$$

$$L_{1l} \leq L_1 \leq L_{1u} \tag{6}$$

$$L_{2l} \leq L_2 \leq L_{2u} \tag{7}$$

In model (1–7), the following terms are defined in Daudin (1992):

$$r = \sqrt{n_1 + n_2}$$

$$c = \sqrt{\frac{n_1 + n_2}{n_2}}$$

$$z = \sqrt{\frac{n_1(x - \mu)}{\sigma}}$$

$$\delta = \frac{(\mu_0 - \mu_1)}{\sigma}$$

$$I_2 = \left[\mu_0 - \frac{L\sigma}{\sqrt{n_1}}, \mu_0 - \frac{L_1\sigma}{\sqrt{n_1}}\right] \cup \left[\mu_0 + \frac{L_1\sigma}{\sqrt{n_1}}, \mu_0 + \frac{L\sigma}{\sqrt{n_1}}\right]$$

$$I_2^* = [-L + \delta\sqrt{n_1}, -L_1 + \delta\sqrt{n_1}) \cup (L_1 + \delta\sqrt{n_1}, L + \delta\sqrt{n_1}]$$

$\Phi(\bullet)$ = standard normal distribution function

$\varphi(\bullet)$ = standard normal density function

One can see from objective function (1) that the expected average sample size is minimized when there is no shift in the process mean, i.e. when $\mu = \mu_0$. The reason for such a choice is that the process is supposed to be under control most of the time.

In constraint (5), L_1 is the lower bound on L. As suggested by Daudin (1992), L must be higher than the classical values 3 or 3.09. Therefore, L_l in constraint (5) can take 3 or 3.09. L_{1l}, L_{1u} in constraint (6) and L_{2l} and L_{2u} in constraint (7) are the lower and upper bounds for L_1 and L_2, respectively. Daudin (1992) suggested that L_1 should be lower than the classical value for a warning limit, which is often set up between 1.3 and 2.0. Therefore, in constraints (6) and (7), L_{1l} and L_{2l} can take 1.3, and L_{1u} and L_{2u} can take 2.

Note that model (1–7) is flexible in setting up the limits L, L_1 and L_2. Based on the needs of the users, any ranges can be set up for limits L, L_1 and L_2 in the formulation for the optimization purpose.

One can see that the minimization problem formulated by model (1–7) is characterized by mixed continuous-discrete variables, and a discontinuous and non-convex solution space. Therefore, if standard non-linear programming techniques are used for solving this type of optimization problem, they will be inefficient and computationally expensive. GAs are well suited for solving such problems and in most cases they find a global optimum solution with a high probability (Rao 1996). Another motivation in using GAs to solve model (1–7) is that since the pioneering work of Holland (1975), GAs have been developed into a general and robust method for solving all kinds of optimization problems (e.g. Goldberg 1989, Potgieter and Stander 1998, Pham and Pham 1999, Vinterbo and Ohno-Machado 2000) and computing software for applications of GAs have been commercially available in the market. Owing to its adaptive nature and flexibility in solving the optimization problems, a GA for solving the DS X-bar chart design problem can be easily extended into solving a sampling chart with more than two stages. Thus, solving model (1–7) using a GA provides a practical way for real-time statistical process control implementation.

3.2. *Implementation of GA*

The tool used for implementing the GAs in solving model (1–7) is a commercially available GA software called Evolver (Palisade 1998). Evolver is used as an add-in program to the Microsoft Excel spreadsheet application. The optimization model (1–7) is set up in an Excel spreadsheet and solved by the GA in Evolver. The operation of the GA involves following steps: (1) create a random initial solution; (2) evaluate fitness, i.e. the objective function that minimizes the average sample size when the process is in control; (3) reproduction and mutation; and (4) generate new solutions.

The quality of the solutions generated by the GA depends on the set-up of its parameters such as population size, crossover and mutation probability. During the implementation, these parameters are set up to obtain the best results.

Crossover probability determines how often crossover will be performed. If there is no crossover, an offspring (new solution) will be an exact copy of the parents (old solutions). If there is a crossover, the offspring is made from parts of the parents' chromosome. If the crossover probability is 100%, then a crossover makes the offspring. Crossover is made in the hope that new chromosomes will have good parts of old chromosomes and maybe the new chromosomes will be better. However, it is good to leave some part of population survive to next generation.

Mutation probability determines how often parts of the chromosomes will be mutated. If there is no mutation, offspring will be taken after crossover (or copy) without any change. If the mutation probability is 100%, the whole chromosome is changed. Mutation is made to prevent the search by the GA falling into local extremes, but it should not occur very often because then the GA will in fact change to a random search.

Each member of the randomly created population is the set of the design variables of the optimization problem (candidate solutions). The population size tells how many organisms (or complete sets of variables) should be stored in the memory at any given time. Although there is still much debate and research about the optimal population size, the common view is that a larger population takes longer to settle on a solution, but it is more likely to find a global solution because of its more diverse gene poll.

There is no general procedure developed for setting up the design parameters. It seems there is no definite relationship between the selected parameters and the optimization results. That is why the several combinations of the GA parameters were tried before the final combination of the population size, crossover probability and mutation probability selected is believed to be the optimal.

In our computational experiment, the population size is set up to 1000. The crossover probability is set up to 0.5 and mutation probability to 0.06.

To verify the performance of the GA in solving model (1–7), the results obtained by the GA were compared with those of Daudin (1992). The results obtained by the GA are no worse than those obtained by DAP. However, the GA provides a more efficient and systematic way to construct the DS X-bar control chart and is ready to be extended to sampling an X-chart with more than two stages. Next, the development of a sampling X-bar control chart with three stages is presented.

4. Development of TS X-bar control charts

Up to today, no effort on the development of methods for constructing TS X-bar control charts has been reported in the literature. Although DS X-bar chats have provided better solutions to agile manufacturing than traditional Shewhart charts, greater efficiency may be achieved if TS charts can be developed. In this section, the TS control procedure, design formulation and efficiency of TS charts in comparison with DS charts are provided.

4.1. *TS X-bar control procedure*

For the TS procedure proposed here, it is assumed that the three successive samples can be taken without any intervening time and therefore they come from the same probability distribution. The design of a TS X-bar control chart involves determining the following parameters.

- $n_1 =$ Sample size of the first sample.
- $n_2 =$ Sample size of the second sample.
- $n_3 =$ Sample size of the third sample.
- $L, L_1 =$ limits at the first stage.
- $L_2, L_3 =$ limits at the second stage.
- $L_4 =$ limits at the third stage.

The following procedure is proposed.

(1) Take an initial sample of size n1. Calculate the sample mean $\overline{X}_1$.

(2) If $\dfrac{\overline{X}_1 - \mu_0}{\sigma/\sqrt{n_1}}$ lies in the range $[-L_1, L_1]$, the process is in control.

(3) If $\dfrac{\overline{X}_1 - \mu_0}{\sigma/\sqrt{n_1}}$ lies in the range $(-\infty, -L]$ or $[L, +\infty)$, the process is out of control.

(4) If $\dfrac{\overline{X}_1 - \mu_0}{\sigma/\sqrt{n_1}}$ lies in the range $(-L, -L_1)$ or (L_1, L), take a second sample of size n_2 and calculate the sample mean $\overline{X}_2$

(5) Calculate the total sample mean $\overline{Y} = \dfrac{n_1\overline{X}_1 + n_2\overline{X}_2}{n_1 + n_2}$ at the second stage.

(6) If $-L < \dfrac{\overline{X}_1 - \mu_0}{\sigma/\sqrt{n_1}} < -L_1$ or $L_1 < \dfrac{\overline{X}_1 - \mu_0}{\sigma/\sqrt{n_1}} < L$, and if $\dfrac{\overline{Y} - \mu_0}{\sigma/\sqrt{n_1 + n_2}} < -L_3$ or $\dfrac{\overline{Y} - \mu_0}{\sigma/\sqrt{n_1 + n_2}} > L_3$, then the process is out of control.

(7) If $-L < \dfrac{\overline{X}_1 - \mu_0}{\sigma/\sqrt{n_1}} < -L_1$ or $L_1 < \dfrac{\overline{X}_1 - \mu_0}{\sigma/\sqrt{n_1}} < L$, and $-L_2 < \dfrac{\overline{Y} - \mu_0}{\sigma/\sqrt{n_1 + n_2}} < L_2$, then the process is in control.

(8) If $-L < \dfrac{\overline{X}_1 - \mu_0}{\sigma/\sqrt{n_1}} < -L_1$ or $L_1 < \dfrac{\overline{X}_1 - \mu_0}{\sigma/\sqrt{n_1}} < L$, and if $-L_3 < \dfrac{\overline{Y} - \mu_0}{\sigma/\sqrt{n_1 + n_2}} < -L_2$ or $L_2 < \dfrac{\overline{Y} - \mu_0}{\sigma/\sqrt{n_1 + n_2}} < L_3$, then take the third sample of size n_3 and calculate the sample mean $\overline{X}_3$.

(9) Calculate the total sample mean $\overline{W} = \dfrac{n_1\overline{X}_1 + n_2\overline{X}_2 + n_3\overline{X}_3}{n_1 + n_2 + n_3}$ at the third stage.

(10) If $-L < \dfrac{\overline{X}_1 - \mu_0}{\sigma/\sqrt{n_1}} < -L_1$ or $L_1 < \dfrac{\overline{X}_1 - \mu_0}{\sigma/\sqrt{n_1}} < L$, and if $-L_3 < \dfrac{\overline{Y} - \mu_0}{\sigma/\sqrt{n_1 + n_2}} < -L_2$ or $L_2 < \dfrac{\overline{Y} - \mu_0}{\sigma/\sqrt{n_1 + n_2}} < L_3$, and if $\dfrac{\overline{W} - \mu_0}{\sigma/\sqrt{n_1 + n_2 + n_3}} < -L_4$ or $\dfrac{\overline{W} - \mu_0}{\sigma/\sqrt{n_1 + n_2 + n_3}} > L_4$, then the process is out of control; else the process is in control.

A TS X-bar chart is given in figure 3.

4.2. *Properties of the TS X-bar control chart*

In developing a TS X-bar control chart, the same assumptions made in the DS X-bar chart are used here. It is assumed that the sample means are independent and normally distributed. Further, it is assumed throughout that when a process is out of control, it is due to a shift in the mean from μ_0 to μ_1, and the process σ remains constant. The shift in process mean is measured by $\delta = (\mu_0 - \mu_1)/\sigma$.

The following intervals are defined thus.

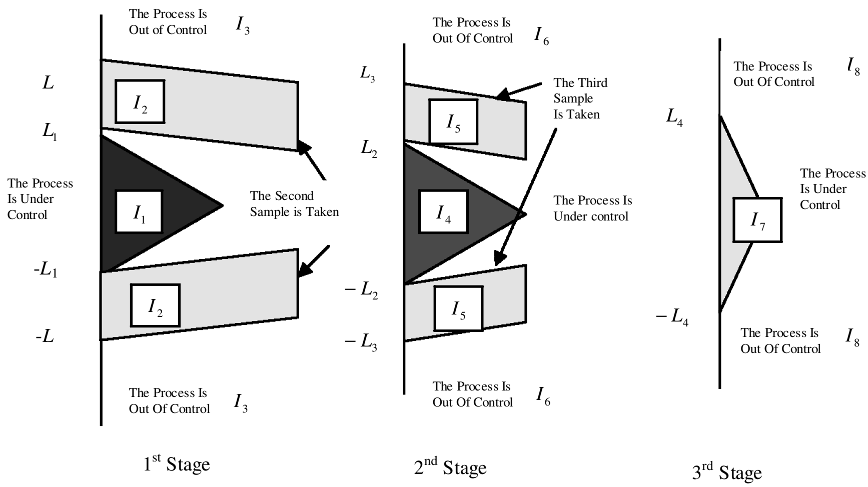

Figure 3. Graphic view of the TS X-bar control chart.

$$I_1 = \left[\mu_0 - \frac{L_1\sigma}{\sqrt{n_1}}, \mu_0 + \frac{L_1\sigma}{\sqrt{n_1}}\right]$$

$$I_2 = \left[\mu_0 - \frac{L\sigma}{\sqrt{n_1}}, \mu_0 - \frac{L_1\sigma}{\sqrt{n_1}}\right] \cup \left[\mu_0 + \frac{L_1\sigma}{\sqrt{n_1}}, \mu_0 + \frac{L\sigma}{\sqrt{n_1}}\right]$$

$$I_3 = \left(-\infty, \mu_0 - \frac{L\sigma}{\sqrt{n_1}}\right] \cup \left[\mu_0 + \frac{L\sigma}{\sqrt{n_1}}, +\infty\right)$$

$$I_4 = \left[\mu_0 - \frac{L_2\sigma}{\sqrt{n_1+n_2}}, \mu_0 + \frac{L_2\sigma}{\sqrt{n_1+n_2}}\right]$$

$$I_5 = \left[\mu_0 - \frac{L_3\sigma}{\sqrt{n_1+n_2}}, \mu_0 - \frac{L_2\sigma}{\sqrt{n_1+n_2}}\right]$$

$$\cup \left[\mu_0 + \frac{L_2\sigma}{\sqrt{n_1+n_2}}, \mu_0 + \frac{L_3\sigma}{\sqrt{n_1+n_2}}\right]$$

$$I_6 = \left(-\infty, \mu_0 - \frac{L_3\sigma}{\sqrt{n_1+n_2}}\right] \cup \left[\mu_0 + \frac{L_3\sigma}{\sqrt{n_1+n_2}}, +\infty\right)$$

$$I_7 = \left[\mu_0 - \frac{L_4\sigma}{\sqrt{n_1+n_2+n_3}}, \mu_0 + \frac{L_4\sigma}{\sqrt{n_1+n_2+n_3}}\right]$$

$$I_8 = \left(-\infty, \mu_0 - \frac{L_4\sigma}{\sqrt{n_1+n_2+n_3}}\right] \cup \left[\mu_0 + \frac{L_4\sigma}{\sqrt{n_1+n_2+n_3}}, +\infty\right).$$

Let P_a be the probability of deciding that the process is in control and P_{ai} be the probability of deciding that process is in control at stage i. Thus, $P_a = P_{a1} + P_{a2} + P_{a3}$. The computation for probabilities P_{a1}, P_{a2} and P_{a3} are explained below.

The probability that the process is in control at the first stage, P_{a1}, is equal to the probability that the sample mean $\overline{X_1}$ is in interval I_1 and can be computed in equation (10):

$$P_{a1} = \Pr[\overline{X_1} \in I_1] = \Phi(L_1 + \delta\sqrt{n_1}) - \Phi(-L_1 + \delta\sqrt{n_1}). \tag{10}$$

The probability that the process is in control at the second stage is equal to the probability that the mean of the first two samples is in interval I_4 given that the mean of the first sample is in the interval I_2. According to the Central Limit Theorem, irrespective of the shape of the distribution of a universe, the distribution of average values, $\overline{X}$'s, of subgroups of size n, drawn from that universe will tend toward a normal distribution. Thus, the random variable $\overline{X}$ has a normal distribution. The probability that $\overline{X_1}$ will be in interval I_2 is an integral over the given interval:

$$P_{a2} = \int_{x \in I_2} \Pr[\overline{Y} \in I_4 | \overline{X_1} = x] f(x)\, \mathrm{d}x, \tag{11}$$

where $f(x)$ is the probability density function of $\overline{X_1}$.

Accordingly, the probability that process is in control at third stage is equal to the probability that the total samples mean, $\overline{W}$, is in interval I_7, given that the mean of the first and second samples, $\overline{Y}$, is in interval I_5, and the mean of the first sample, $\overline{X_1}$, is in interval I_2:

$$P_{a3} = \int_{x \in I_2} \int_{y \in I_5} \Pr[\overline{W} \in I_7 | \overline{Y} = y \text{ and } \overline{X_1} = x] f(y) f(x)\, \mathrm{d}y\, \mathrm{d}x, \tag{12}$$

where $f(y)$ is the probability density function of $\overline{Y}$ and $f(x)$ the probability density function of $\overline{X_1}$.

Let $P_a(\mu = \mu_0)$ be the P_a obtained by setting $\mu = \mu_0$ (no shift in the process mean). Then $1 - P_a(\mu = \mu_0)$ represents the probability of concluding that the process is out of control when it is in control, i.e. $\Pr[\text{out of control} | \mu = \mu_0] = 1 - P_a(\mu = \mu_0)$. Accordingly, $P_a(\mu = \mu_1)$ represents the probability of concluding that the process is in control when it is out of control, i.e. $\Pr[\text{ in control} | \mu = \mu_1] = P_a(\mu = \mu_1)$.

The expected average sample size for a TS X-bar control chart when there is no shift in process mean is computed as:

$$E(N) = n_1 + n_2 \Pr[\overline{X_1} \in I_2 | \mu = \mu_0] + n_3 \Pr[\overline{Y} \in I_5 | \mu = \mu_0], \tag{13}$$

where $\Pr[\overline{X_1} \in I_2 | \mu = \mu_0]$ is the probability of taking the second sample when there is no shift in process mean, which can be computed as following:

$$\begin{aligned} \Pr[\overline{X_1} \in I_2 | \mu = \mu_0] = {} & \Phi(L + \delta\sqrt{n_1}) - \Phi(L_1 + \delta\sqrt{n_1}) \\ & + \Phi(-L_1 + \delta\sqrt{n_1}) - \Phi(-L + \delta\sqrt{n_1}). \end{aligned} \tag{14}$$

$\Pr[\overline{Y} \in I_5 | \mu = \mu_0]$ in equation (13) is the probability of taking the third sample and could be expressed as:

$$\Pr[\overline{Y} \in I_5 | \mu = \mu_0] = Pr[\overline{Y} \in I_5 | \overline{X_1} \in I_2] = \int_{x \in I_2} \Pr[\overline{Y} \in I_5 | \overline{X_1} = x] f(x)\, \mathrm{d}x. \tag{15}$$

The right-hand sides of equations (11), (12) and (15) can be computed using numerical integration. For more details on these calculations, see appendix A.

4.3. *TS X-bar chart design formulation*

To solve the design problem for the TS X-bar control chart, it is formulated as an optimization problem:

$$\underset{n_1,n_2,n_3,L,L_1,L_2,L_3,L_4}{\text{Min}} \quad n_1 +_2 \Pr[\overline{X_1} \in I_2 | \mu = \mu_0] + n_3 \Pr[\overline{Y} \in I_5 | \mu = \mu_0], \tag{16}$$

subject to

$$\Pr[\text{out of control}|\mu = \mu_0] = 1 - P_a(\mu = \mu_0) \le \alpha, \text{i.e.}$$

$$\begin{aligned}
&\{\Phi[L_1] - \Phi[-L_1]\} - \int_{z\in I_2^*} \left\{\Phi\left[cL_2 - z\sqrt{\frac{n_1}{n_2}}\right] - \Phi\left[-cL_2 - z\sqrt{\frac{n_1}{n_2}}\right]\varphi(z)\,\mathrm{d}z\right\} \\
&- \int_{z_1\in I_2^*}\int_{z_2\in I_5^*}\left[\int \Phi\left(cL_4 - \sqrt{\frac{n_1}{n_3}}z_1 - \sqrt{\frac{n_2}{n_3}}z_2\right)\right. \\
&\left. - \Phi\left(-cL_4 - \sqrt{\frac{n_1}{n_3}}z_1 - \sqrt{\frac{n_2}{n_3}}z_2\right)\right]\varphi(z_2)\varphi(z_1)\,\mathrm{d}z_2\,\mathrm{d}z_1 \le \alpha.
\end{aligned} \tag{17}$$

For $\Pr[\text{in control}|\mu = \mu_1] = P_a(\mu = \mu_1)\beta$, i.e.

$$\begin{aligned}
&\{\Phi[L_1 + \delta\sqrt{n_1}] - \Phi[-L_1 + \delta\sqrt{n_1}]\} \\
&\quad + \int_{z\in I_2}\left\{\Phi\left[cL_2 + rc\delta - z\sqrt{\frac{n_1}{n_2}}\right] - \Phi\left[-cL_2 + rc\delta - z\sqrt{\frac{n_1}{n_2}}\right]\right\}\varphi(z)\,\mathrm{d}z \\
&\quad \times \int_{z_1\in I_2^*}\int_{z_2\in I_5^*}\left[\Phi\left(cL_4 + rc\delta - \sqrt{\frac{n_1}{n_3}}z_1 - \sqrt{\frac{n_2}{n_3}}z_2\right)\right. \\
&\quad \left. - \Phi\left(-cL_4 + rc\delta - \sqrt{\frac{n_1}{n_3}}z_1 - \sqrt{\frac{n_2}{n_3}}z_2\right)\right]\varphi(z_2)\varphi(z_1)\,\mathrm{d}z_2\,\mathrm{d}z_1 \\
&\quad \le \beta.
\end{aligned} \tag{18}$$

The objective function (16) in this optimization model is to minimize the average sample size subject to two constraints. Constraint (17) ensures that the probability of making a false alarm is $\le \alpha$ (Type I error) and constraint (18) ensures that the probability of not detecting a shift in process mean is $\le \beta$ (Type II error). In addition to constraints (17) and (18), the lower and upper bounds are imposed on L, L_1, L_2, L_3 and L_4. The lower and upper bounds are set up in the same way as in the DS X-bar chart design. In addition, integer constraints are imposed on n_1, n_2 and n_3.

4.4. *Comparison of TS X-bar chart with DS X-bar chart*

Model (16–18) for the TS chart design was solved by the GA with the same population size, crossover probability rate and mutation probability rate as in solving model (1–7) for the DS chart.

The average sample size of the TS charts for the same ARL and δ was compared with that of the DS charts. The results of the comparison are given in table 1, where the TS X-bar control uses fewer samples on average than the DS X-bar control at all levels of shift in process mean.

The set-up of the experiment comparing the TS and DS control charts is explained thus.

- ARL_0, ARL_1 and the shift in the process mean, δ, were set up the same way as in the experiment comparing the DS X-bar chart and the Shewhart X-bar chart by Daudin (1992).

Chart type	ARL_0	$ARL_1(\delta)$	n_1	n_2	n_3	L_1	L	L_2	L_3	L_4	$E(N)$	Decease %
DS	370.4	1.186(2.83)	1	2	–	1.81	5.0	2.77	–	–	1.14	1.8
TS			1	1	1	1.62	3.07	1.8	3.35	2.86	1.12	
DS	500.0	1.222(2.83)	1	2	–	1.90	5.0	2.85	–	–	1.12	3.6
TS			1	1	1	1.79	3.0	1.8	3.01	2.93	1.08	
DS	370.4	1.186(2)	2	3	–	1.74	5.0	2.85	–	–	2.25	7.6
TS			2	1	1	1.76	3.00	1.8	3.69	2.66	2.08	
DS	500.0	1.222(2)	2	3	–	1.82	5.0	2.94	–	–	2.21	2.7
TS			2	2	1	1.80	3.00	1.8	3.39	2.85	2.15	
DS	370.4	1.186(1.79)	2	4	–	1.37	5.0	2.90	–	–	2.68	13.8
TS			2	2	1	1.47	3.00	1.8	3.3	2.87	2.31	
DS	500.0	1.222(1.79)	2	5	–	1.45	5.0	2.98	–	–	2.59	9.3
TS			2	2	2	1.49	3.00	1.47	4.51	2.81	2.35	
DS	370.4	1.186(1.63)	2	6	–	1.22	5.0	2.87	–	–	3.33	20.7
TS			2	2	3	1.23	3.32	1.55	3.90	2.97	2.64	
DS	500.0	1.222(1.63)	2	6	–	1.31	5.0	2.96	–	–	3.14	16.6
TS			2	2	3	1.34	3.67	1.56	3.14	2.88	2.62	
DS	370.4	1.186(1.51)	3	6	–	1.61	5.0	2.86	–	–	3.76	10.6
TS			3	3	2	1.57	3.00	1.8	3.61	2.81	3.36	
DS	500.0	1.222(1.51)	3	6	–	1.61	5.0	2.96	–	–	3.64	9.3
TS			3	3	2	1.66	3.0	1.8	3.86	2.87	3.30	
DS	370.0	1.186(1.41)	3	7	–	1.32	5.0	2.88	–	–	4.30	3.0
TS			3	3	4	1.41	3.0	1.61	4.07	2.86	4.17	
DS	500.0	1.222(1.41)	3	7	–	1.41	5.0	2.97	–	–	4.11	11.7
TS			3	3	5	1.48	3.17	1.8	3.44	2.89	3.63	
DS	370.4	1.186(1.33)	4	7	–	1.54	5.0	2.88	–	–	4.87	8.2
TS			4	4	3	1.63	3.00	1.66	3149	2.84	4.47	
DS	500.0	1.222(1.33)	3	9	–	1.31	5.0	2.95	–	–	4.72	18.9
TS			3	3	4	1.32	3.72	1.68	3.56	2.82	3.83	
DS	370.4	1.186(1.26)	4	9	–	1.44	5.0	2.86	–	–	5.53	10.8
TS			4	4	6	1.49	3.13	1.78	3.09	2.91	4.77	
DS	500.0	1.222(1.26)	4	9	–	1.53	5.0	2.95	–	–	5.14	10.1
TS			4	5	4	1.59	3.0	1.8	3.39	2.97	4.62	
DS	370.4	1.186(1.15)	5	10	–	1.47	5.0	2.87	–	–	6.42	12.8
TS			5	8	3	1.55	3.0	1.8	3.57	2.89	5.60	
DS	500.0	1.222(1.15)	5	10	–	1.55	5.0	2.96	–	–	6.21	5.0
TS			5	5	6	1.43	3.36	18	3.81	2.98	5.90	
DS	370.0	1.186(0.89)	8	17	–	1.41	5.0	2.88	–	–	10.69	11.3
TS			8	10	5	1.49	3	1.67	3.18	2.72	9.48	
DS	500.0	1.222(0.89)	8	18	–	1.53	5.0	2.95	–	–	10.28	15.56
TS			8	5	7	1.54	3.085	1.71	3.74	2.76	8.68	

Table 1. Results of a comparison between the TS and DS charts.

- The same α and β constraints were imposed on solving the optimization problems for both the DS and TS control schemes. It means that both control chart schemes will satisfy the manufacturers' and customers' requirements.
- The expected average sample size (E (N)) was calculated for the DS and TS schemes by solving the optimization problems using the GA.
- The decreased percentage of the expected average sample size was calculated using:

$$\text{Decrease} = \left(\frac{E(N)_{DS} - E(N)_{TS}}{E(N)_{DS}} \right) \times 100\%.$$

As one might see from the comparison procedure, the goal of the comparison was to determine how large sample size is needed for each of the control chart schemes in order to detect a shift of certain magnitude, given that the Type I and II error requirements for both charts are the same.

Comparison of the TS and DS control charts shows that for all the shifts in the process mean, the expected average sample size for the TS control chart scheme is less than that of the DS control chart scheme. The decrease varies from 1.8 to 18.0%. However, no pattern in the percentage decrease of the expected average sample size could be generalized as the shift in the process mean decreases. One explanation for such randomness is that in the experiments an equal amount of time was set up to run the GA for finding the design parameters for each process mean shift. It is known that the convergence of the GA solutions depends on the size of the problem and its computational times. Owing to the nature of the non-linear objective function and constraints and the mixed real and integer variables of the optimization problems, in some cases the GA found solutions that were very close to the optimum. Indeed, the purpose of this paper was to show the statistical advantage of the TS scheme over the DS scheme. For further improvements of the results, the increase in computational time for running the GA could be considered.

5. Conclusions

As today's manufacturing firms are moving towards agile manufacturing, quick and economic on-line statistical process control solutions are in high demand. Multiple sampling X-bar control charts are such an alternative. They can be designed to allow quick detection of a small shift in the process mean and provide a quick response in an agile manufacturing environment.

Daudin's (1992) method of optimizing a DS X-bar chart has a distinct advantage of giving an earlier signal of a shift in the process mean, which is of great importance in an agile manufacturing environment. However, it requires quite complicated calculations and cannot be extended easily to more than two multiple-sampling stages.

In this paper, the design of the DS (DS) X-bar control charts were formulated and solved with a GA. Based on the results of solving the DS chart design problems, TS X-bar control chars were developed. The efficiency of the TS charts was compared with that of the DS charts, the results showing that TS charts were more efficient in terms of minimizing the average sample size. There is a decrease in the expected average sample size for the TS control scheme compared with the DS scheme. However, the TS scheme requires more time for decision-making than the DS scheme since the number of the sampling stage is larger. For cases where the statistical efficiency is more important than the time required for decision-making, the TS control scheme could be considered as a good alternative to traditional control charts.

The results presented in this paper can be further extended to develop methods for the design of multiple-sampling X-bar charts with more than three sampling stages. The development of a DS control chart scheme for monitoring the process variance could also be a future research topic.

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Appendix

1. The computation of (11), i.e. the probability that the process is in control at stage two, is derived in Daudin (1992) as following:

$$\mathrm{P}_{a2} = \int_{z \in I_2^*} \left[\Phi\left(L_2 c + rc\delta - \frac{\sqrt{n_1}}{\sqrt{n_2}} z \right) - \Phi\left(-L_2 c + rc\delta - \frac{\sqrt{n_1}}{\sqrt{n_2}} z \right) \right] \varphi(z)\, \mathrm{d}z.$$

where

$$z = \sqrt{n_1}(x-\mu)/\sigma; \quad r = \sqrt{n_1+n_2}; \quad c = \sqrt{\frac{n_1+n_2}{n_2}}$$

2. The computation of (15), i.e. the probability that $\overline{Y}$ is in interval I_5, can be derived as following:

$$\begin{aligned}\Pr[\overline{Y} \in I_5] = \Pr&\left[\frac{L_2\sigma}{\sqrt{n_1+n_2}} \le \frac{n_1\overline{X_1}+n_2\overline{X_2}}{n_1+n_2} - \mu_0 \le \frac{L_3\sigma}{\sqrt{n_1+n_2}}\right] \\ &+ \Pr\left[\frac{-L_3\sigma}{\sqrt{n_1+n_2}} \le \frac{n_1\overline{X_1}+n_2\overline{X_2}}{n_1+n_2} - \mu_0 \le \frac{-L_2\sigma}{\sqrt{n_1+n_2}}\right].\end{aligned}$$

Let $r = \sqrt{n_1+n_2}$, $Z_1 = \sqrt{n_1}(\overline{X_1}-\mu)/\sigma$, and $Z_2 = \sqrt{n_2}(\overline{X_2}-\mu)/\sigma$, then:

$$\begin{aligned}&\Pr\left[\frac{L_2\sigma}{\sqrt{n_1+n_2}} \le \frac{n_1\overline{X_1}+n_2\overline{X_2}}{n_1+n_2} - \mu_0 \le \frac{L_3\sigma}{\sqrt{n_1+n_2}}\right] \\ &= \Pr\left[\frac{L_2\sigma}{r\sigma} \le \frac{n_1(\overline{X_1}-\mu)}{(n_1+n_2)\sigma} + \frac{n_2(\overline{X_2}-\mu)}{(n_1+n_2)\sigma} - \delta \le \frac{L_3\sigma}{r\sigma}\right] \\ &= \Pr\left[\frac{L_2}{r} \le \frac{\sqrt{n_1}\sqrt{n_1}(\overline{X_1}-\mu)}{(n_1+n_2)\sigma} + \frac{\sqrt{n_2}\sqrt{n_2}(\overline{X_2}-\mu)}{(n_1+n_2)\sigma} - \delta \le \frac{L_3}{r}\right] \\ &= \Pr\left[\frac{L_2}{r} \le \frac{\sqrt{n_1}Z_1}{n_1+n_2} + \frac{\sqrt{n_2}Z_2}{n_1+n_2} - \delta \le \frac{L_3}{r}\right] \\ &= \Pr\left[\frac{L_2}{r} + \delta \le \frac{\sqrt{n_2}}{n_1+n_2}\left(\frac{\sqrt{n_1}}{\sqrt{n_2}} Z_1 + Z_2\right) \le \frac{L_3}{r} + \delta\right] \\ &= \Pr\left[\frac{L_2+r\delta}{r} \le \sqrt{\frac{n_2}{n_1+n_2}}\frac{1}{\sqrt{n_1+n_2}}\left(\frac{\sqrt{n_1}}{\sqrt{n_2}} Z_1 + Z_2\right) \le \frac{L_3+r\delta}{r}\right] \\ &= \Pr\left[L_2 + r\delta \le \sqrt{\frac{n_2}{n_1+n_2}}\left(\frac{\sqrt{n_1}}{\sqrt{n_2}} Z_1 + Z_2\right) \le L_3 + r\delta\right] \\ &= \Pr\left[L_2 c + rc\delta \le \frac{\sqrt{n_1}}{\sqrt{n_2}} Z_1 + Z_2 \le L_3 c + rc\delta\right]\end{aligned}$$

Let $z = \sqrt{n_1}(x-\mu)/\sigma$, then:

$$\begin{aligned}&\Pr\left[\frac{L_2\sigma}{\sqrt{n_1+n_2}} \le \frac{n_1\overline{X_1}+n_2\overline{X_2}}{n_1+n_2} - \mu_0 \le \frac{L_3\sigma}{\sqrt{n_1+n_2}}\right] \\ &= \Pr\left[L_2 c + rc\delta - \frac{\sqrt{n_1}}{\sqrt{n_2}} z \le Z_2 \le L_3 c + rc\delta - \frac{\sqrt{n_1}}{\sqrt{n_2}} z\right].\end{aligned}$$

In a similar way:

$$\Pr\left[\frac{-L_3\sigma}{\sqrt{n_1+n_2}} \le \frac{n_1\overline{X_1}+n_2\overline{X_2}}{n_1+n_2} - \mu_0 \le \frac{-L_2\sigma}{\sqrt{n_1+n_2}}\right]$$

$$= \Pr\left[-L_3c + rc\delta - \frac{\sqrt{n_1}}{\sqrt{n_2}}\,z \le Z_2 \le -L_2c + rc\delta - \frac{\sqrt{n_1}}{\sqrt{n_2}}\,z\right]$$

Therefore:

$$Pr[\overline{Y} \in I_5] = \int_{z\in I_2^*}\left[\Phi\left(cL_3 + rc\delta - \sqrt{\frac{n_1}{n_2}}z\right) - \Phi(cL_2 + rc\delta - \sqrt{\frac{n_1}{n_2}}z\right)\right]\varphi(z)\,\mathrm{d}z$$

$$\int_{z\in I_2^*}\left[\Phi\left(-cL_2 + rc\delta - \sqrt{\frac{n_1}{n_2}}z\right) - \Phi\left(-cL_3 + rc\delta - \sqrt{\frac{n_1}{n_2}}z\right)\right]\varphi(z)\,\mathrm{d}z.$$

When $\mu = \mu_0, \delta = 0.$

Thus,

$$\Pr[\overline{Y} \in I_5|\mu = \mu_0] = \int_{z\in I_2^*}\left[\Phi\left(cL_3 - \sqrt{\frac{n_1}{n_2}}z\right) - \Phi\left(cL_2 - \sqrt{\frac{n_1}{n_2}}z\right)\right]\varphi(z)\,\mathrm{d}z$$
$$+\int_{z\in I_2^*}\left[\Phi\left(-cL_2 - \sqrt{\frac{n_1}{n_2}}z\right) - \Phi\left(-cL_3 - \sqrt{\frac{n_1}{n_2}}z\right)\right]\varphi(z)\,\mathrm{d}z$$

3. The computation of (12), i.e. the probability that process is in control at the third stage, can be derived as following:

$$P_{a3} = \int_{x\in I_2}\int_{y\in I_5}\Pr[\overline{W} \in I_7 \big| \overline{Y} = y \quad \text{and} \quad \overline{X_1} = x]f(y)f(x)\,\mathrm{d}y\,\mathrm{d}x$$

Let

$$r_2 = \sqrt{n_1+n_2+n_3}, Z_1 = \sqrt{n_1}(\overline{X_1}-\mu)/\sigma, Z_2 = \sqrt{n_2}(\overline{X_2}-\mu)/\sigma Z_3 = \sqrt{n_3}(\overline{X_3}-\mu)/\sigma,$$

and

$$c_2 = \sqrt{\frac{n_1+n_2+n_3}{n_3}},$$

then:

$$\Pr[\overline{W} \in I_7] = \Pr\left[\left|\frac{n_1\overline{X_1}+n_2\overline{X_2}+n_3\overline{X_3}}{n_1+n_2+n_3} - \mu_0\right| < \frac{L_4\sigma}{\sqrt{n_1+n_2+n_3}}\right]$$

$$= \Pr\left[\frac{-L_4\sigma}{r} < \frac{n_1(\overline{X_1}-\mu)+n_2(\overline{X_2}-\mu)+n_3(\overline{X_3}-\mu)}{n_1+n_2+n_3} - (\mu-\mu_0) \le \frac{L_4\sigma}{r}\right]$$

$$= \Pr\left[\frac{-L_4\sigma}{r_2\sigma} \le \frac{n_1(\overline{X_1}-\mu)}{(n_1+n_2+n_3)\sigma} + \frac{n_2(\overline{X_2}-\mu)}{(n_1+n_2+n_3)\sigma}\right.$$

$$+\frac{n_3(\overline{X_3}-\mu)}{(n_1+n_2+n_3)\sigma}-\delta \leq \frac{L_4\sigma}{r_2\sigma}\Bigg]$$

$$= \Pr\left[\frac{-L_4}{r_2} \leq \frac{\sqrt{n_1}\sqrt{n_1}(\overline{X_1}-\mu)}{(n_1+n_2+n_3)\sigma}+\frac{\sqrt{n_2}\sqrt{n_2}(\overline{X_2}-\mu)}{(n_1+n_2+n_3)\sigma}\right.$$

$$\left.+\frac{\sqrt{n_3}\sqrt{n_3}(\overline{X_3}-\mu)}{(n_1+n_2+n_3)\sigma}-\delta \leq \frac{L_4}{r_2}\right]$$

$$= \Pr\left[\frac{-L_4}{r_2} \leq \frac{\sqrt{n_1}Z_1}{n_1+n_2+n_3}+\frac{\sqrt{n_2}Z_2}{n_1+n_2+n_3}+\frac{\sqrt{n_3}Z_3}{n_1+n_2+n_3}-\delta \leq \frac{L_4}{r_2}\right]$$

$$= \Pr\left[\frac{-L_4}{r_2}+\delta \leq \frac{\sqrt{n_3}}{n_1+n_2+n_3}\left(\frac{\sqrt{n_1}}{\sqrt{n_3}}Z_1+\frac{\sqrt{n_2}}{\sqrt{n_3}}Z_2+Z_3\right) \leq \frac{L_4}{r_2}+\delta\right]$$

$$= \Pr\left[\frac{-L_4+r\delta}{r_2} \leq \sqrt{\frac{n_3}{n_1+n_2+n_3}}\frac{1}{\sqrt{n_1+n_2+n_3}}\right.$$

$$\left.\times\left(\sqrt{\frac{n_1}{\sqrt{n_3}}}Z_1+\frac{\sqrt{n_2}}{\sqrt{n_3}}Z_2+Z_3\right) \leq \frac{L_4+r\delta}{r_2}\right]$$

$$= \Pr\left[-L_4+r_2\delta \leq \sqrt{\frac{n_3}{n_1+n_2+n_3}}\left(\frac{\sqrt{n_1}}{\sqrt{n_3}}Z_1+\frac{\sqrt{n_2}}{\sqrt{n_3}}Z_2+Z_3\right) \leq L_4+r_2\delta\right]$$

$$= \Pr -L_4\, c_2 + r_2\, c_2\, \delta \leq \frac{\sqrt{n_1}}{\sqrt{n_3}}Z_1+\frac{\sqrt{n_2}}{\sqrt{n_3}}Z_2+Z_3 \leq L_4\, c_2 + r_2\, c_2\, \delta\Bigg]$$

$$= \Pr\left[-L_4c_2+r_2c_2\delta-\frac{\sqrt{n_1}}{\sqrt{n_3}}Z_1-\frac{\sqrt{n_2}}{\sqrt{n_3}}Z_2 \leq Z_3 \leq L_4c_2+r_2c_2\delta\right.$$

$$\left.-\frac{\sqrt{n_1}}{\sqrt{n_3}}Z_1-\frac{\sqrt{n_2}}{\sqrt{n_3}}Z_2\right].$$

Let $z_1=\sqrt{n_1}(x_1-\mu)/\sigma$ and $z_2=\sqrt{n_2}(x_2-\mu)/\sigma$, then:

$$\Pr[\overline{W}\in I_7|\overline{Y}=y \quad \text{and} \quad \overline{X_1}=x] = P[\overline{W}\in I_7|Z_2=z_2 \quad \text{and} \quad Z_1=z_1]$$

$$= \Pr\left[-L_4c_2+r_2c_2\delta-\frac{\sqrt{n_1}}{\sqrt{n_3}}z_1-\frac{\sqrt{n_2}}{\sqrt{n_3}}z_2 \leq Z_3 \leq L_4c_2+rc\delta-\frac{\sqrt{n_1}}{\sqrt{n_3}}z_1-\frac{\sqrt{n_2}}{\sqrt{n_3}}z_2\right]$$

$$= P_{a3} = \int_{z_1\in I_2^*}\int_{z_2\in I_5^*}\Pr[\overline{W}\in I_7|Z_2=z_2 \quad \text{and} \quad Z_1=z_1]\varphi(z_2)\varphi(z_1)\,\mathrm{d}z_2\,\mathrm{d}z_1$$

$$= \int_{z_1\in I_2^*}\int_{z_2\in I_5^*}\left[\left(\Phi\left(cL_4+rc\delta-\sqrt{\frac{n_1}{n_3}}z_1-\sqrt{\frac{n_2}{n_3}}z_2\right)\right.\right.$$

$$\left.\left.-\Phi\left(-cL_4+rc\delta-\sqrt{\frac{n_1}{n_3}}z_1-\sqrt{\frac{n_2}{n_3}}z_2\right)\right)\right]\varphi(z_2)\,\mathrm{d}z_2\varphi(z_1)\,\mathrm{d}z_1,$$

where $I_5^*=[-L_3+\delta\sqrt{n_1+n_2},-L_2+\delta\sqrt{n_1+n_2}]\cup[L_2+\delta\sqrt{n_1+n_2},L_3+\delta\sqrt{n_1+n_2}]$

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